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State Space Model for Autopilot Design of Aerospace Vehicles

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ABSTRACT

This report is a follow on to the report given in DSTO-TN-0449 and considers the derivation of the mathematical model for aerospace vehicles and missile autopilots in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the non-linearities are "structured" (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques, particularly for autopilot design of aerospace vehicles executing high g-manoevres.

This report also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. The models developed are useful for further research on precision optimum guidance and control. It is hoped that the model will provide more accurate presentations of missile autopilot dynamics and will be used for adaptive and integrated guidance & control of agile missiles.

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Executive Summary

Requirements for next generation guided weapons and other aerospace vehicles, particularly with respect to their capability to engage high speed, highly agile targets and achieve precision end-game trajectory, have prompted a revision of the way in which the guidance and autopilot design is undertaken. This report considers the derivation of the mathematical models for aerospace vehicles and missile autopilots in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the non-linearities are “structured” (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques.

This report which is a follow on report to DSTO-TN-0449, also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. The models developed are useful for further research on precision optimum guidance and control. It is hoped that the model will provide more accurate presentations of aerospace vehicles autopilot dynamics and will be used for adaptive and integrated guidance & control of agile missiles and other aerospace vehicles that do not necessarily have symmetric cruciform airframes.

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1. Introduction

Requirements for next generation guided weapons, particularly with respect to their capability to engage high speed, highly agile targets and achieve precision end-game trajectory, have prompted a revision of the way in which the guidance and autopilot design are undertaken. Integrating the guidance and control function is a synthesis approach that is being pursued as it allows the optimisation of the overall system performance. This approach requires a more complete representation of the airframe dynamics and the guidance system. The use of state space model allows the application of modern control techniques such as the optimal adaptive control and parameter estimation techniques [10] to be utilised. In this report we derive the autopilot model that will serve as a basis for an adaptive autopilot design and allow further extension of this to integrated guidance and control system design.

Over the years a number of authors [1-3, 6-9] have considered modelling, analysis and design of autopilots for atmospheric flight vehicles including guided missiles. In the majority of the published work on autopilot analysis and design, locally linearised versions of the model with decoupled airframe dynamics have been considered. This latter simplification arises out of the assumption that the airframe and its mass distribution are symmetrical about the body axes, and that the yaw, pitch and roll motion about the equilibrium state remain “small”. As a result, many of the autopilot analysis and design techniques, considered in open literature, use classical control approach, such as: single input/single output transfer-functions characterisation of the system dynamics, Bode, Nyquist, root-locus and transient -response analysis and synthesis techniques [5, 7]. These techniques are valid for a limited set of flight regimes and their extension to cover a wider set of flight regimes and airframe configurations requires autopilot gain and compensation network switching.

With the advent of fast processors it is now possible to take a more integrated approach to autopilot design. Modern optimal control techniques allow the designer to consider autopilots with high-order dynamics (large number of states) with multiple inputs/outputs and to synthesise controllers such that the error between the demanded and the achieved output is minimised. Moreover, with real-time mechanisation any changes in the airframe aerodynamics can be identified (parameter estimation) and compensated for by adaptively varying the optimum control gain matrix. This approach should lead to missile systems that are able to execute high g -manoeuvres (required by modern guided weapons), adaptively adjust control parameters (to cater for widely varying flight profiles) as well as account for non-symmetric airframe and mass distributions. Typically, for a missile autopilot, the input is the demanded control surface deflection and outputs are the achieved airframe (lateral) accelerations and body rates measured about the body axes. Other input/output variables (such as: the flight path angle and angle rate or the body angles) can be derived directly from lateral accelerations and body rates.

This report considers the derivation of the mathematical model for a missile autopilot in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the nonlinearities are “structured” (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program

and possibly for future application of non-linear analysis and synthesis techniques. Detailed consideration of the quadratic/bilinear type of dynamic systems is given in [4].

This report which is a follow on report from the previous report [1,2], also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. For sake of clarity, Figure 2.1 is a symmetric cruciform missile, however the models developed are valid for non axis-symmetric aerospace vehicles. Tables A-1.1 to A-1.3 contain the various aerodynamic derivatives and coefficients.

2. State Space Aerodynamics Model

The airframe, actuation and sensor measurement equations have been derived in detail in Appendix A in this section we give the main results that will be used for matrix based computation of the state space model.

2.1 Nonlinear Airframe Model

Conventions and notations for vehicle body axes systems as well as the forces, moments and other quantities are shown in Figure 2.1 and defined in Table 2.1.

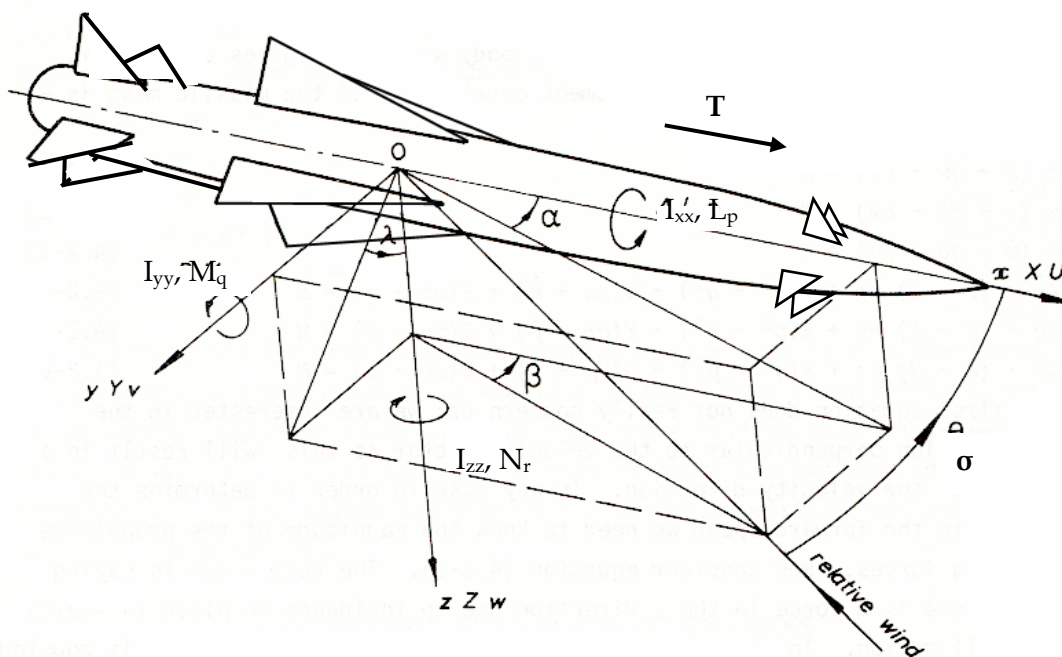


Figure 2.1 Motion variable notations

The variables shown in Figure 2.1 are defined as:

m - mass of a vehicle.

α - incidence in the pitch plane.

β - incidence in the yaw plane.

λ - incidence plane angle.

σ - total incidence, such that: $\tan \alpha = \tan \sigma \cos \lambda$, and $\tan \beta = \tan \sigma \sin \lambda$.

T - thrust.

Table 2.1: Motion variables

Vehicle Body Axes System	Roll axis	Pitch axis	Yaw axis
Angular rates	p	q	r
Component of vehicle velocity along each axis	u	v	w
Component of aerodynamic forces acting on vehicle along each axis	X	Y	Z
Moments acting on vehicle about each axis	L	M	N
Moments of inertia about each axis	I_{xx}	I_{yy}	I_{zz}
Products of each inertia	I_{yz}	I_{zx}	I_{xy}
Longitudinal and lateral accelerations	a_x	a_y	a_z
Euler angles	ϕ	θ	ψ
Gravity along each axis	g_x	g_y	g_z
Vehicle thrust along the body axis	T		

Tail Control Configuration:

We shall use the following notation: ξ - aileron deflection; η - elevator deflection; ζ - rudder deflection. Figure 2.2 defines the control surface convention. Here the control surfaces are numbered as shown and the deflections ($\delta_1, \delta_2, \delta_3, \delta_4$) are taken to be positive if clockwise, looking outwards along the individual hinge axis. Thus, Aileron deflection:

$\xi = \frac{1}{4}(\delta_1 + \delta_2 + \delta_3 + \delta_4)$, if all four control surfaces are active; or $\xi = \frac{1}{2}(\delta_1 + \delta_3)$, or

$\xi = \frac{1}{2}(\delta_2 + \delta_4)$ if only two surfaces are active. Positive control deflection (ξ) causes negative

roll. Elevator deflection: $\eta = \frac{1}{2}(\delta_1 - \delta_3)$. Positive control deflection (η) causes negative pitch.

Rudder deflection: $\zeta = \frac{1}{2}(\delta_2 - \delta_4)$. Positive control deflection (ζ) causes negative yaw.

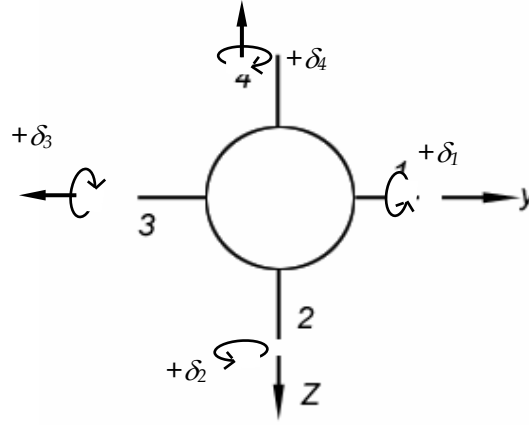


Figure 2.2 Control surfaces seen from the rear of a missile

Canard Control Configuration:

For a canard configuration, same convention will be used for control surface deflections; however, it is noted that the force and moment coefficients will have opposite signs. Canard control is generally not used for roll control.

Euler Equations of Motion

The six equations of motion for a body with six degrees of freedom may be written as [1-3]:

$$m(\dot{u} + wq - vr) = X + T + g_x m \quad (2-1)$$

$$m(\dot{v} + ur - wp) = Y + g_y m \quad (2-2)$$

$$m(\dot{w} - uq + vp) = Z + g_z m \quad (2-3)$$

$$I_{xx}\dot{p} - (I_{yy} - I_{zz})qr + I_{yz}(r^2 - q^2) - I_{zx}(pq + \dot{r}) + I_{xy}(rp - \dot{q}) = L \quad (2-4)$$

$$I_{yy}\dot{q} - (I_{zz} - I_{xx})rp + I_{zx}(p^2 - r^2) - I_{xy}(qr + \dot{p}) + I_{yz}(pq - \dot{r}) = M \quad (2-5)$$

$$I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xy}(q^2 - p^2) - I_{yz}(rp + \dot{q}) + I_{zx}(qr - \dot{p}) = N \quad (2-6)$$

Here $(\dot{\cdot}) = \frac{d}{dt}$ - is the derivative operator.

Based on the Euler equations above, the nonlinear (quadratic) airframe state space model is given by (see equations A-1.1 to A-1.5):

$$\frac{d}{dt} \underline{x}_3^{[I]} = [F_0] \underline{x}_3^{[2]} + [G_0] \underline{u}_3^{[I]} + \underline{g}^{[I]} \quad (2-7)$$

Where:

$$\underline{x}_3^{[I]} = \begin{bmatrix} \underline{x}_1^{[I]} \\ \underline{x}_2^{[I]} \end{bmatrix} = [u \quad v \quad w \quad / \quad p \quad q \quad r]^T : \text{is a } 6 \times 1 \text{ linear-state vector.}$$

$$\underline{x}_3^{[2]} = \begin{bmatrix} \underline{x}_1^{[2]} \\ \underline{x}_2^{[2]} \end{bmatrix} = [uq \quad ur \quad vp \quad vr \quad wp \quad wq \quad / \quad p^2 \quad pq \quad pr \quad q^2 \quad qr \quad r^2]^T : \text{is a } 12 \times 1 \text{ quadratic-state vector.}$$

$$\underline{u}_3^{[I]} = \begin{bmatrix} \underline{u}_1^{[I]} \\ \underline{u}_2^{[I]} \end{bmatrix} = [\tilde{X} + \tilde{T} \quad \tilde{Y} \quad \tilde{Z} \quad / \quad L \quad M \quad N]^T : \text{is } 6 \times 1 \text{ a vector function of control inputs, forces and moments.}$$

$$\underline{g}^{[I]} = \begin{bmatrix} \underline{g} \\ \underline{0}_{3 \times 1} \end{bmatrix} = [g_x \quad g_y \quad g_z \quad 0 \quad 0 \quad 0]^T : \text{is the } 6 \times 1 \text{ gravity (or disturbance) vector.}$$

$$[F_0] = \begin{bmatrix} [C_0] & / & [0_{3 \times 6}] \\ \text{---} & / & \text{---} \\ [0_{3 \times 6}] & / & [A_0]^{-I} [B_0] \end{bmatrix} : \text{is a } 6 \times 12 \text{ state-coefficient matrix.}$$

$$[G_0] = \begin{bmatrix} [I_{3 \times 3}] & / & [0_{3 \times 3}] \\ \text{---} & / & \text{---} \\ [0_{3 \times 3}] & / & [A_0]^{-I} \end{bmatrix} : \text{is a } 6 \times 6 \text{ input-coefficient matrix.}$$

$$[A_0] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{zx} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{yz} & I_{zz} \end{bmatrix} : \text{is a } 3 \times 3 \text{ matrix.}$$

$$[B_0] = \begin{bmatrix} 0 & I_{zx} & -I_{xy} & I_{yz} & (I_{yy} - I_{zz}) & -I_{yz} \\ -I_{zx} & -I_{yz} & (I_{zz} - I_{xx}) & 0 & I_{xy} & I_{zx} \\ I_{xy} & (I_{xx} - I_{yy}) & I_{yz} & -I_{xy} & -I_{zx} & 0 \end{bmatrix} : \text{is a } 3 \times 6 \text{ matrix.}$$

$$[C_0] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} : \text{is a } 3 \times 6 \text{ matrix.}$$

Subscripts under $[I]$ and $[0]$ matrices denote the matrix dimensions. Generally, not all state variables in the state equation are accessible or measurable. The vehicle angular rate components (roll rate p , pitch rate q , and yaw rate r) and the acceleration components (a_x , a_y , a_z) are commonly available and can be measured using the IMU.

2.2 Body Acceleration Model

Equations for the vehicle actual body accelerations and body rates are derived in Appendix A equations (A-1.6) to (A-1.11). These equations allow position offsets from c.g for observing the body accelerations. The acceleration components at point O (where O is at a distance of d_x , d_y and d_z from the central point of gravity, c.g., along x-, y- and z-axis, respectively), may be written as:

$$\begin{aligned} a_x &= \dot{u} + qw - rv - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \\ &= \tilde{X} + \tilde{T} + g_x - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \end{aligned} \quad (2-8)$$

$$\begin{aligned} a_y &= \dot{v} + ru - pw + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \\ &= \tilde{Y} + g_y + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \end{aligned} \quad (2-8)$$

$$\begin{aligned} a_z &= \dot{w} + pv - qu + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2) \\ &= \tilde{Z} + g_z + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2) \end{aligned} \quad (2-9)$$

After some matrix manipulations, the body acceleration model may be written as (see equations (A-1.6) to (A-1.11):

$$\underline{y}_3^{[I]} = [J_0] \underline{x}_3^{[I]} + [K_0] \underline{x}_3^{[2]} + [L_0] \underline{u}_3^{[I]} + [M_0] \underline{g}^{[I]} \quad (2-10)$$

Where:

$$\underline{y}_1^{[I]} = [a_x \quad a_y \quad a_z]^T : \text{is a } 3 \times 1 \text{ body acceleration vector.}$$

$$\underline{y}_2^{[I]} = \underline{x}_2^{[I]} = [p \quad q \quad r]^T : \text{is a } 3 \times 1 \text{ body rate vector}$$

$$\underline{y}_3^{[I]} = [\underline{y}_1^{[I]} \quad / \quad \underline{y}_2^{[I]}]^T : \text{is a } 6 \times 1 \text{ output vector.}$$

$$[D_0] = \begin{bmatrix} 1 & 0 & 0 & 0 & d_z & -d_y \\ 0 & 1 & 0 & -d_z & 0 & d_x \\ 0 & 0 & 1 & d_y & -d_x & 0 \end{bmatrix} : \text{is a } 3 \times 6 \text{ accelerometer position 'offset' matrix.}$$

$$[D_I] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & d_y & d_z & -d_x & 0 & -d_x \\ 0 & 1 & 0 & 0 & -1 & 0 & -d_y & d_x & 0 & 0 & d_z & -d_y \\ -1 & 0 & 1 & 0 & 0 & 0 & -d_z & 0 & d_x & -d_z & d_y & 0 \end{bmatrix} : \text{is a } 3 \times 12 \text{ accelerometer position 'offset' matrix.}$$

$$[J_0] = \begin{bmatrix} [0_{3 \times 6}] \\ \hline [0_{3 \times 3} \quad / \quad I_{3 \times 3}] \end{bmatrix} : \text{is a } 6 \times 6 \text{ linear-state output coefficient matrix.}$$

$$[K_0] = \begin{bmatrix} [D_0 F_0 + D_I] \\ \hline [0_{3 \times 12}] \end{bmatrix} : \text{is a } 6 \times 12 \text{ quadratic-state output coefficient matrix.}$$

$$[L_0] = \begin{bmatrix} [D_0 G_0] \\ \hline [0_{3 \times 6}] \end{bmatrix} : \text{is a } 6 \times 6 \text{ coefficient matrix.}$$

$$[M_0] = \begin{bmatrix} [D_0] \\ \hline [0_{3 \times 6}] \end{bmatrix} : \text{is a } 6 \times 6 \text{ output coefficient matrix.}$$

Note: equation (2-10) represents the actual accelerations and body rates outputs; these have to be measured using a body fixed IMU (accelerometers and gyros).

2.3 Accelerometer Dynamics Model

The dynamic model for the accelerometers is derived in Appendix A, equations (A-1.12) to (A-1.17). A second order linear dynamics is assumed for the accelerometer model.

$$\frac{d}{dt} \underline{x}_4^{[I]} = [W_0] \underline{x}_4^{[I]} + [W_2] \underline{x}_3^{[2]} + [W_3] \underline{u}_3^{[I]} + [W_4] \underline{g}^{[I]} \quad (2-11)$$

Where:

$$\underline{x}_4^{[I]} = \begin{bmatrix} a_{x0} & \dot{a}_{x0} & a_{y0} & \dot{a}_{y0} & a_{z0} & \dot{a}_{z0} \end{bmatrix}^T : \text{is a } 6 \times 1 \text{ accelerometer state vector.}$$

$$[W_0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_x^2 & -2\zeta_x\omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_y^2 & -2\zeta_y\omega_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & -2\zeta_z\omega_z \end{bmatrix} : \text{ is a } 6 \times 6 \text{ coefficient matrix}$$

containing accelerometer parameters.

$$[W_I] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_z^2 \end{bmatrix} : \text{ is a } 6 \times 3 \text{ coefficient matrix containing accelerometer parameters.}$$

$$[W_2] = [W_I D_0 F_0 + W_I D_I] : \text{ is a } 6 \times 12 \text{ coefficient matrix.}$$

$$[W_3] = [W_I D_0 G_0] : \text{ is a } 6 \times 6 \text{ coefficient matrix.}$$

$$[W_4] = [W_I D_0] : \text{ is a } 6 \times 6 \text{ coefficient matrix.}$$

The accelerometer measurement model is given by (see equation (A-1.18)):

$$\underline{z}_4^{[I]} = [J_I] \underline{x}_4^{[I]} + \underline{v}_{ab}^{[I]} + \underline{v}_{ad}^{[I]} + \underline{v}_{as}^{[I]} + \underline{v}_{an}^{[I]} \quad (2-12)$$

Where:

$$\underline{z}_4^{[I]} = \begin{bmatrix} a_{xm} & a_{ym} & a_{zm} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ accelerometer measurement vector.}$$

$$\underline{v}_{ab}^{[I]} = \begin{bmatrix} v_{xb} & v_{yb} & v_{zb} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ accelerometer bias error vector.}$$

$$\underline{v}_{ad}^{[I]} = \begin{bmatrix} v_{xd} & v_{yd} & v_{zd} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ accelerometer drift error vector.}$$

$$\underline{v}_{as}^{[I]} = \begin{bmatrix} v_{xs} & v_{ys} & v_{zs} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ accelerometer scale factor error vector.}$$

$$\underline{v}_{an}^{[I]} = \begin{bmatrix} v_{xn} & v_{yn} & v_{zn} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ accelerometer noise error vector.}$$

$$[J_I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} : \text{is a } 3 \times 6 \text{ matrix.}$$

2.4 Gyro Dynamics Model

The dynamic model for the gyros is derived in Appendix A, equations (A-1.19) to (A-1.21). A second order linear dynamics is assumed for the gyro model.

$$\frac{d}{dt} \underline{x}_5^{[I]} = [W_5] \underline{x}_5^{[I]} + [W_7] \underline{x}_3^{[I]} \quad (2-13)$$

Where:

$$\underline{x}_5^{[I]} = \begin{bmatrix} p_o & \dot{p}_o & q_o & \dot{q}_o & r_o & \dot{r}_o \end{bmatrix}^T : \text{is a } 6 \times 1 \text{ gyro state vector.}$$

$$[W_5] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_p^2 & -2\zeta_p \omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_q^2 & -2\zeta_q \omega_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_r^2 & -2\zeta_r \omega_r \end{bmatrix} : \text{is a } 6 \times 6 \text{ coefficient matrix}$$

containing gyro parameters.

$$[W_6] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_q^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 \end{bmatrix} : \text{is a } 6 \times 3 \text{ coefficient matrix containing gyro parameters.}$$

$$[W_7] = [0_{6 \times 3} \quad / \quad W_6] : \text{is a } 6 \times 6 \text{ coefficient matrix.}$$

The gyro measurement model is given by (see equation (A-1.22)):

$$\underline{z}_5^{[I]} = [J_2] \underline{x}_5^{[I]} + \underline{v}_{gb}^{[I]} + \underline{v}_{gd}^{[I]} + \underline{v}_{gs}^{[I]} + \underline{v}_{gn}^{[I]} \quad (2-14)$$

Where:

$\underline{z}_4^{[I]} = \begin{bmatrix} p_m & q_m & r_m \end{bmatrix}^T$: is a 3x1 gyro measurement vector.

$\underline{v}_{gb}^{[I]} = \begin{bmatrix} v_{pb} & v_{qb} & v_{rb} \end{bmatrix}^T$: is a 3x1 gyro bias error vector.

$\underline{v}_{gd}^{[I]} = \begin{bmatrix} v_{pd} & v_{qd} & v_{rd} \end{bmatrix}^T$: is a 3x1 gyro drift error vector.

$\underline{v}_{gs}^{[I]} = \begin{bmatrix} v_{ps} & v_{qs} & v_{rs} \end{bmatrix}^T$: is a 3x1 gyro scale factor error vector.

$\underline{v}_{gn}^{[I]} = \begin{bmatrix} v_{pn} & v_{qn} & v_{rn} \end{bmatrix}^T$: is a 3x1 gyro noise error vector.

$[J_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$: is a 3x6 matrix.

2.5 Actuation Servo Model

The dynamic model for the actuation system is derived in Appendix A, equations (A-1.23) to (A-1.28). A second order linear dynamics is assumed for the gyro model.

This equation is of the form:

$$\frac{d}{dt} \underline{x}_6^{[I]} = [V_0] \underline{x}_6^{[I]} + [V_I] \underline{u}_4^{[I]} \quad (2-15)$$

Where:

$\underline{x}_6^{[I]} = \begin{bmatrix} \xi_o & \dot{\xi}_o & \eta_o & \dot{\eta}_o & \zeta_o & \dot{\zeta}_o \end{bmatrix}^T$: is a 6x1 state vector.

$\underline{u}_4^{[I]} = \underline{\alpha}_i = \begin{bmatrix} \xi_i & \eta_i & \zeta_i \end{bmatrix}^T$: is a 3x1 control (servo actuator) input vector.

$$[V_o] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_\xi^2 & -2\zeta_\xi\omega_\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_\eta^2 & -2\zeta_\eta\omega_\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_\zeta^2 & -2\zeta_\zeta\omega_\zeta \end{bmatrix} : \text{is a 6x6 servo actuator coefficient matrix.}$$

$$[V_I] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_\xi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_\eta^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_\zeta^2 \end{bmatrix} : \text{is a 6x3 servo input coefficient matrix.}$$

If the actuator system noise is included in the model then the actual output from the actuator servo may be written as:

$$\frac{d}{dt} \underline{x}_6^{[I]} = [V_o] \underline{x}_6^{[I]} + [V_I] \underline{u}_4^{[I]} + \underline{v}_{s_n}^{[I]} \quad (2-16)$$

We may also write for the actuator output:

$$\underline{y}_6^{[I]} = [\xi_o \quad \eta_o \quad \varsigma_o]^T = [V_2] \underline{x}_6^{[I]} \quad (2-17)$$

Where:

$$\underline{v}_{s_n}^{[I]} = \begin{bmatrix} v_{\xi_n} & v_{\dot{\xi}_n} & v_{\eta_n} & v_{\dot{\eta}_n} & v_{\zeta_n} & v_{\dot{\zeta}_n} \end{bmatrix}^T : \text{is a 6x1 actuator servo noise error vector.}$$

$$[V_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} : \text{is a 3x6 matrix.}$$

Note that:

$$\begin{aligned} \underline{u}_3^{[I]} &= [\tilde{X}(\cdot) \quad \tilde{Y}(\cdot) \quad \tilde{Z}(\cdot) \quad L(\cdot) \quad M(\cdot) \quad N(\cdot)]^T \\ &= \underline{f}(u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi_o, \eta_o, \varsigma_o) = \underline{f}(\underline{x}_3^{[I]}, \dot{\underline{x}}_3^{[I]}, [V_2] \underline{x}_6^{[I]}) \end{aligned} \quad (2-18)$$

2.6 Overall Nonlinear (Quadratic) Airframe Model including IMU

Equations (A-1.5), (A-1.17), (A-1.21) and (A-1.26) combine to give us an overall airframe, IMU and actuator dynamic model; this equation is of the form:

$$\frac{d}{dt} \underline{x}_7^{[I]} = [F_1] \underline{x}_7^{[I]} + [F_2] \underline{x}_3^{[2]} + [G_1] \underline{u}_3^{[I]}(..) + [G_2] \underline{u}_4^{[I]} + [H_1] \underline{g}^{[I]} + [H_2] \underline{v}_{s_n}^{[I]} \quad (2-19)$$

Where:

$$\underline{x}_7^{[I]} = \left[\underline{x}_3^{[I]T} \quad / \quad \underline{x}_4^{[I]T} \quad / \quad \underline{x}_5^{[I]T} \quad / \quad \underline{x}_6^{[I]T} \right]^T : \text{is a } 24 \times 1 \text{ state vector.}$$

$$[F_1] = \begin{bmatrix} [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ --- & / & --- & / & --- & / & --- \\ [0_{6 \times 6}] & / & [W_0] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ --- & / & --- & / & --- & / & --- \\ [W_7] & / & [0_{6 \times 6}] & / & [W_5] & / & [0_{6 \times 6}] \\ --- & / & --- & / & --- & / & --- \\ [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [V_0] \end{bmatrix} : \text{is a } 24 \times 24 \text{ coefficient matrix.}$$

$$[F_2] = \left[[F_0]^T \quad / \quad [W_2]^T \quad / \quad [0_{12 \times 6}] \quad / \quad [0_{12 \times 6}] \right]^T : \text{is a } 24 \times 12 \text{ coefficient matrix.}$$

$$[G_1] = \left[[G_0]^T \quad / \quad [W_3]^T \quad / \quad [0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \right]^T : \text{is a } 24 \times 6 \text{ coefficient matrix.}$$

$$[G_2] = \left[[0_{3 \times 6}] \quad / \quad [0_{3 \times 6}] \quad / \quad [0_{3 \times 6}] \quad / \quad [V_1]^T \right]^T : \text{is a } 24 \times 3 \text{ coefficient matrix.}$$

$$[H_1] = \left[[I_{6 \times 6}] \quad / \quad [W_4]^T \quad / \quad [0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \right]^T : \text{is a } 24 \times 6 \text{ coefficient matrix.}$$

$$[H_2] = \left[[0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \quad / \quad [I_{6 \times 6}] \right]^T : \text{is a } 24 \times 6 \text{ coefficient matrix.}$$

A block diagram of the decomposed version (derived by considering the sub-matrices) of the overall model is given in Figure A-1.1. See appendix section 3.

2.7 The Measurement Model

Equation (2-12) and (2-14) may be combined to give the overall airframe and IMU (Gyros, Accelerometers) measurement model (see equation (A-1.30)):

$$\underline{z}_7^{[I]} = [J_6] \underline{x}_7^{[I]} + \underline{v}_b^{[I]} + \underline{v}_d^{[I]} + \underline{v}_s^{[I]} + \underline{v}_n^{[I]} \quad (2-20)$$

Where:

$\underline{z}_7^{[I]} = \begin{bmatrix} \underline{z}_4^{[I]T} & / & \underline{z}_5^{[I]T} \end{bmatrix}^T = \begin{bmatrix} a_{x_m} & a_{y_m} & a_{z_m} & p_m & q_m & r_m \end{bmatrix}^T$: is a 6x1 IMU (gyro + accelerometer) measurement vector.

$\underline{v}_b^{[I]} = \begin{bmatrix} \underline{v}_{a_b}^{[I]T} & / & \underline{v}_{g_b}^{[I]T} \end{bmatrix}^T = \begin{bmatrix} v_{x_b} & v_{y_b} & v_{z_b} & v_{p_b} & v_{q_b} & v_{r_b} \end{bmatrix}^T$: is a 6x1 IMU bias error vector.

$\underline{v}_d^{[I]} = \begin{bmatrix} \underline{v}_{a_d}^{[I]T} & / & \underline{v}_{g_d}^{[I]T} \end{bmatrix}^T = \begin{bmatrix} v_{x_d} & v_{y_d} & v_{z_d} & v_{p_d} & v_{q_d} & v_{r_d} \end{bmatrix}^T$: is a 6x1 IMU drift error vector.

$\underline{v}_s^{[I]} = \begin{bmatrix} \underline{v}_{a_s}^{[I]T} & / & \underline{v}_{g_s}^{[I]T} \end{bmatrix}^T = \begin{bmatrix} v_{x_s} & v_{y_s} & v_{z_s} & v_{p_s} & v_{q_s} & v_{r_s} \end{bmatrix}^T$: is a 6x1 IMU scale factor error vector.

$\underline{v}_n^{[I]} = \begin{bmatrix} \underline{v}_{a_n}^{[I]T} & / & \underline{v}_{g_n}^{[I]T} \end{bmatrix}^T = \begin{bmatrix} v_{x_n} & v_{y_n} & v_{z_n} & v_{p_n} & v_{q_n} & v_{r_n} \end{bmatrix}^T$: is a 6x1 IMU noise error vector.

$\underline{J}_6 = \begin{bmatrix} 0_{3 \times 6} & / & J_1 & / & 0_{3 \times 6} & / & 0_{3 \times 6} \\ --- & / & --- & / & --- & / & --- \\ 0_{3 \times 6} & / & 0_{3 \times 6} & / & J_2 & / & 0_{3 \times 6} \end{bmatrix}$: is a 6x24 matrix.

3. Linearised State Space Airframe Model

The linearised state space model is derived in Appendix A, section 2 (equations (A-2.1) to (A-2.9)). It is assumed that \tilde{X} , \tilde{Y} , \tilde{Z} , L , M and N are functions of $u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \dot{\xi}, \dot{\eta}, \dot{\varsigma}$ and first order linearization of these nominal values $u_0, v_0, w_0, p_0, q_0, r_0, \xi_0, \eta_0$ and ς_0 , are considered. The linearised airframe model is given by:

The ‘small’ perturbation model for the quadratic dynamic model of equation (A-1.29) may be written as (see equation (A-2.5)):

$$\begin{aligned} \frac{d}{dt} \Delta \underline{x}_7^{[I]} = & [F_1] \Delta \underline{x}_7^{[I]} + [F_2 E_4 + G_1 E_0] \Delta \underline{x}_3^{[I]} + [G_1 E_2] \Delta \underline{x}_6^{[I]} + [G_1 E_3] \frac{d}{dt} \Delta \underline{x}_3^{[I]} \\ & + [G_2] \Delta \underline{u}_4^{[I]} + [H_2] \Delta \underline{v}_{s_n}^{[I]} \end{aligned} \quad (3-1)$$

This equation will be referred to as the decomposed form (see Appendix A section 3)

Where:

$$[F_2 E_4 + G_1 E_0] = \begin{bmatrix} [F_0 E_4 + G_0 E_0] \\ \text{-----} \\ [W_2 E_4 + W_3 E_0] \\ \text{-----} \\ [0_{6 \times 6}] \\ \text{-----} \\ [0_{6 \times 6}] \end{bmatrix}; [G_1 E_2] = \begin{bmatrix} [G_0 E_2] \\ \text{-----} \\ [W_3 E_2] \\ \text{-----} \\ [0_{6 \times 6}] \\ \text{-----} \\ [0_{6 \times 6}] \end{bmatrix}; [G_1 E_3] = \begin{bmatrix} [G_0 E_3] \\ \text{-----} \\ [W_3 E_3] \\ \text{-----} \\ [0_{6 \times 6}] \\ \text{-----} \\ [0_{6 \times 6}] \end{bmatrix}$$

Equation (3-1) may be written in a compact form as:

$$\frac{d}{dt} \Delta \underline{x}^{[I]} = [F_5] \Delta \underline{x}^{[I]} + [G_3] \Delta \underline{u}^{[I]} + [H_3] \Delta \underline{v}_{s_n}^{[I]} \quad (3-2)$$

Where:

$$[E_0] = \begin{bmatrix} \tilde{X}_u & \tilde{X}_v & \tilde{X}_w & \tilde{X}_p & \tilde{X}_q & \tilde{X}_r \\ \tilde{Y}_u & \tilde{Y}_v & \tilde{Y}_w & \tilde{Y}_p & \tilde{Y}_q & \tilde{Y}_r \\ \tilde{Z}_u & \tilde{Z}_v & \tilde{Z}_w & \tilde{Z}_p & \tilde{Z}_q & \tilde{Z}_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} : \text{is a } 6 \times 6 \text{ aero-derivative matrix.}$$

$$[E_1] = \begin{bmatrix} \tilde{X}_\xi & \tilde{X}_\eta & \tilde{X}_\varsigma \\ \tilde{Y}_\xi & \tilde{Y}_\eta & \tilde{Y}_\varsigma \\ \tilde{Z}_\xi & \tilde{Z}_\eta & \tilde{Z}_\varsigma \\ L_\xi & L_\eta & L_\varsigma \\ M_\xi & M_\eta & M_\varsigma \\ N_\xi & N_\eta & N_\varsigma \end{bmatrix} : \text{is a } 6 \times 3 \text{ control-derivative matrix.}$$

$[E_2] = [E_1 V_2]$: is a 6×6 matrix.

$$[E_3] = \begin{bmatrix} \tilde{X}_{\dot{u}} & \tilde{X}_{\dot{v}} & \tilde{X}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Y}_{\dot{u}} & \tilde{Y}_{\dot{v}} & \tilde{Y}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Z}_{\dot{u}} & \tilde{Z}_{\dot{v}} & \tilde{Z}_{\dot{w}} & 0 & 0 & 0 \\ L_{\dot{u}} & L_{\dot{v}} & L_{\dot{w}} & 0 & 0 & 0 \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & 0 & 0 & 0 \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & 0 & 0 & 0 \end{bmatrix} : \text{is a } 6 \times 6 \text{ aero-derivative matrix.}$$

$$[E_4] = \begin{bmatrix} q_0 & 0 & 0 & 0 & u_0 & 0 \\ r_0 & 0 & 0 & 0 & 0 & u_0 \\ 0 & p_0 & 0 & v_0 & 0 & 0 \\ 0 & r_0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & p_0 & w_0 & 0 & 0 \\ 0 & 0 & q_0 & 0 & w_0 & 0 \\ 0 & 0 & 0 & 2p_0 & 0 & 0 \\ 0 & 0 & 0 & q_0 & p_0 & 0 \\ 0 & 0 & 0 & r_0 & 0 & p_0 \\ 0 & 0 & 0 & 0 & 2q_0 & 0 \\ 0 & 0 & 0 & 0 & r_0 & q_0 \\ 0 & 0 & 0 & 0 & 0 & 2r_0 \end{bmatrix} : \text{ is a } 12 \times 6 \text{ matrix of steady state values.}$$

$$[F_3] = [I_{24 \times 24}] - [G_1 E_3 \quad 0_{24 \times 6} \quad 0_{24 \times 6} \quad 0_{24 \times 6}] : \text{ is a } 24 \times 24 \text{ coefficient matrix.}$$

$$[F_4] = [F_1] + [F_2 E_4 + G_1 E_0 \quad 0_{24 \times 6} \quad 0_{24 \times 6} \quad 0_{24 \times 6}] : \text{ is a } 24 \times 24 \text{ coefficient matrix.}$$

$$[F_5] = [F_3]^{-1} [F_4] : \text{ is a } 24 \times 24 \text{ coefficient matrix.}$$

$$[G_3] = [F_3]^{-1} [G_2] : \text{ is a } 24 \times 3 \text{ coefficient matrix.}$$

$$[H_3] = [F_3]^{-1} [H_2] : \text{ is a } 24 \times 3 \text{ coefficient matrix.}$$

3.1 Linearised Measurement Model

Small perturbation model of the measurement model (see equation (A-1.30)) may be written as:

$$\underline{\Delta x}_7^{[I]} = [J_6] \underline{\Delta x}_7^{[I]} + \underline{\Delta v}_b^{[I]} + \underline{\Delta v}_b^{[I]} + \underline{\Delta v}_s^{[I]} + \underline{\Delta v}_n^{[I]} \quad (3-3)$$

Δ : denotes small perturbation about nominal values

Finally, linearising the output equation (2-10) (see also equation (A-1.11)) gives us:

$$\underline{\Delta y}_3^{[I]} = [J_7] \underline{\Delta x}_3^{[I]} + [L_0] \underline{\Delta u}_3^{[I]} \quad (3-4)$$

Where:

$$[J_7] = [J_0 + K_0 E_4] : \text{ is a } 6 \times 6 \text{ coefficient matrix.}$$

4. Conclusions

Both the non-linear and linearised autopilot models have been derived in this report. The state-space model of a missile autopilot needs to be validated by comparing the model with the other published results, and through both open and closed-loop systems simulation. The non-linear dynamics model presented as structural quadratic algebraic system is novel and will be used for developing non-linear control techniques suitable for missile systems high g-manoevuvres and operating of a range of aerodynamics conditions. The models developed in this report are useful for applications to precision optimum and adaptive guidance and control. It is hoped that the higher order model with motion and inertial coupling will provide more accurate representation of autopilot dynamics particularly for non axis-symmetric airframes that could be used for adaptive and integrated guidance & control of missiles such as air breathing supersonic and hypersonic vehicles.

5. References

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Appendix A:

A.1. Non-linear (Quadratic) Airframe and IMU Dynamics

Equations (2-1) to (2-3) represent the force equations of a generalised rigid body and describe the translational motion of its centre of gravity (c.g) since the origin of the vehicle body axes is assumed to be co-located with the body c.g. Equations (2-4) to (2-6) represent the moment equations of a generalised rigid body and describe the rotational motion about the body axes through its c.g. Separating the derivative terms and after some algebraic manipulation, Equations (2-1) to (2-3) may be written in a vector form as:

$$\frac{d}{dt} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} uq \\ ur \\ vp \\ vr \\ wp \\ wq \end{bmatrix} + \begin{bmatrix} \tilde{X} + \tilde{T} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} + \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \quad (\text{A-1.1})$$

Where:

$$\tilde{X} = \frac{X}{m}; \quad \tilde{Y} = \frac{Y}{m}; \quad \tilde{Z} = \frac{Z}{m}; \quad \tilde{T} = \frac{T}{m}.$$

Note: that in the above equations, the states (u, v, w, p, q, r) appear as a quadratic form expression.

In matrix notation equations (2-4) to (2-6) (section.2) may be written as:

$$[A_0] \frac{d}{dt} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = [B_0] \begin{bmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \quad (\text{A-1.2})$$

Here again, the states (p, q, r) appear as a quadratic form expression.

Where:

$$[A_0] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{zx} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{yz} & I_{zz} \end{bmatrix} : \text{is a } 3 \times 3 \text{ matrix.}$$

$$[B_0] = \begin{bmatrix} 0 & I_{zx} & -I_{xy} & I_{yz} & (I_{yy} - I_{zz}) & -I_{yz} \\ -I_{zx} & -I_{yz} & (I_{zz} - I_{xx}) & 0 & I_{xy} & I_{zx} \\ I_{xy} & (I_{xx} - I_{yy}) & I_{yz} & -I_{xy} & -I_{zx} & 0 \end{bmatrix} : \text{is a } 3 \times 6 \text{ matrix.}$$

Equation (A-1.2) may also be written as:

$$\frac{d}{dt} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = [A_0]^{-1} [B_0] \begin{bmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{bmatrix} + [A_0]^{-1} \begin{bmatrix} L \\ M \\ N \end{bmatrix} \quad (\text{A-1.3})$$

Where:

$$[A_0]^{-1} = \frac{1}{\Delta} \begin{bmatrix} (I_{yy}I_{zz} - I_{yz}^2) & (I_{zz}I_{xy} + I_{yz}I_{zx}) & (I_{yz}I_{xy} + I_{yy}I_{zx}) \\ (I_{zz}I_{xy} + I_{yz}I_{zx}) & (I_{xx}I_{zz} - I_{zx}^2) & (I_{xx}I_{yz} + I_{zx}I_{xy}) \\ (I_{yz}I_{xy} + I_{yy}I_{zx}) & (I_{xx}I_{yz} + I_{xy}I_{zx}) & (I_{xx}I_{yy} - I_{xy}^2) \end{bmatrix} : \text{a } 3 \times 3 \text{ matrix.}$$

$$\Delta = (I_{xx}I_{yy}I_{zz} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2 - 2I_{yz}I_{zx}I_{xy}).$$

The selection of the particular order of the terms in the ‘quadratic-state’ vectors $[uq \ ur \ vp \ vr \ wp \ wq]^T$ of Equation (A-1.1) and $[p^2 \ pq \ pr \ q^2 \ qr \ r^2]^T$ of Equation (A-1.2) is made on the basis of retaining lexicographic order of the variables. Combining Equations (A-1.1) and (A-1.3), we obtain the full 6th order rigid body dynamics state equations as:

$$\frac{d}{dt} \begin{bmatrix} \underline{x}_1^{[I]} \\ \underline{x}_2^{[I]} \end{bmatrix} = \begin{bmatrix} [C_0] & / & [0_{3 \times 6}] \\ [0_{3 \times 6}] & / & [A_0]^{-1}[B_0] \end{bmatrix} \begin{bmatrix} \underline{x}_1^{[2]} \\ \underline{x}_2^{[2]} \end{bmatrix} + \begin{bmatrix} [I_{3 \times 3}] & / & [0_{3 \times 3}] \\ [0_{3 \times 3}] & / & [A_0]^{-1} \end{bmatrix} \begin{bmatrix} \underline{u}_1^{[I]} \\ \underline{u}_2^{[I]} \end{bmatrix} + \begin{bmatrix} \underline{g} \\ \underline{0} \end{bmatrix} \quad (\text{A-1.4})$$

Where:

$$[C_0] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} : \text{is a } 3 \times 6 \text{ matrix.}$$

$$\underline{x}_1^{[1]} = [u \quad v \quad w]^T : \text{is a } 3 \times 1 \text{ linear state vector.}$$

$$\underline{x}_2^{[1]} = [p \quad q \quad r]^T : \text{is a } 3 \times 1 \text{ linear state vector.}$$

$$\underline{x}_1^{[2]} = [uq \quad ur \quad vp \quad vr \quad wp \quad wq]^T : \text{is a } 6 \times 1 \text{ quadratic state vector.}$$

$$\underline{x}_2^{[2]} = [p^2 \quad pq \quad pr \quad q^2 \quad qr \quad r^2]^T : \text{is a } 6 \times 1 \text{ quadratic state vector.}$$

$$\underline{u}_1^{[1]} = [\tilde{X} + \tilde{T} \quad \tilde{Y} \quad \tilde{Z}]^T : \text{is a } 3 \times 1 \text{ 'force' input vector.}$$

$$\underline{u}_2^{[1]} = [L \quad M \quad N]^T : \text{is a } 3 \times 1 \text{ 'moment' input vector.}$$

$$\underline{g} = [g_x \quad g_y \quad g_z]^T : \text{is a } 3 \times 1 \text{ gravity vector.}$$

Note: superscript ^[1] is used to indicate that the state vector is linear; while superscript ^[2] is used to indicate that the state vector is a quadratic/bilinear form.

Equation (A-1.4) may be written in a compact form as:

$$\frac{d}{dt} \underline{x}_3^{[1]} = [F_0] \underline{x}_3^{[2]} + [G_0] \underline{u}_3^{[1]} + \underline{g}^{[1]} \quad (\text{A-1.5})$$

Where:

$$\underline{x}_3^{[1]} = \begin{bmatrix} \underline{x}_1^{[1]} \\ - \\ \underline{x}_2^{[1]} \end{bmatrix} = [u \quad v \quad w \quad / \quad p \quad q \quad r]^T : \text{is a } 6 \times 1 \text{ linear-state vector.}$$

$\underline{x}_3^{[2]} = \begin{bmatrix} \underline{x}_1^{[2]} \\ \underline{x}_2^{[2]} \end{bmatrix} = \begin{bmatrix} uq & ur & vp & vr & wp & wq & / & p^2 & pq & pr & q^2 & qr & r^2 \end{bmatrix}^T$: is a 12x1 quadratic-state vector.

$\underline{u}_3^{[I]} = \begin{bmatrix} \underline{u}_1^{[I]} \\ \underline{u}_2^{[I]} \end{bmatrix} = \begin{bmatrix} \tilde{X} + \tilde{T} & \tilde{Y} & \tilde{Z} & / & L & M & N \end{bmatrix}^T$: is 6x1 a vector function of control inputs, forces and moments.

$\underline{g}^{[I]} = \begin{bmatrix} \underline{g} \\ \underline{0}_{3 \times 1} \end{bmatrix} = \begin{bmatrix} g_x & g_y & g_z & 0 & 0 & 0 \end{bmatrix}^T$: is the 6x1 gravity (or disturbance) vector.

$[F_0] = \begin{bmatrix} [C_0] & / & [0_{3 \times 6}] \\ \text{---} & / & \text{---} \\ [0_{3 \times 6}] & / & [A_0]^{-1}[B_0] \end{bmatrix}$: is a 6x12 state-coefficient matrix.

$[G_0] = \begin{bmatrix} [I_{3 \times 3}] & / & [0_{3 \times 3}] \\ \text{---} & / & \text{---} \\ [0_{3 \times 3}] & / & [A_0]^{-1} \end{bmatrix}$: is a 6x6 input-coefficient matrix.

Subscripts under $[I]$ and $[0]$ matrices denote the matrix dimensions.

A.1.1 Body Acceleration Model

Generally, not all state variables in the state equation are accessible or measurable. The common accessible measurement variables, in most missiles or airplanes, are the angular rate components (roll rate p , pitch rate q , and yaw rate r) and the acceleration components (a_x , a_y , a_z). The acceleration components at point O (where O is at a distance of d_x , d_y and d_z from the central point of gravity, c.g., along x-, y- and z-axis, respectively), may be written as [11]:

$$\begin{aligned} a_x &= \dot{u} + qw - rv - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \\ &= \tilde{X} + \tilde{T} + g_x - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \end{aligned} \quad (\text{A-1.6})$$

$$\begin{aligned} a_y &= \dot{v} + ru - pw + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \\ &= \tilde{Y} + g_y + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \end{aligned} \quad (\text{A-1.7})$$

$$\begin{aligned}
a_z &= \dot{w} + pv - qu + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2) \\
&= \tilde{Z} + g_z + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2)
\end{aligned} \tag{A-1.8}$$

Or in matrix form: \rightarrow

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \underline{y}_I^{[I]} = [D_0] \frac{d}{dt} \begin{bmatrix} \underline{x}_I^{[I]} \\ - \\ \underline{x}_2^{[I]} \end{bmatrix} + [D_I] \begin{bmatrix} \underline{x}_I^{[2]} \\ - \\ \underline{x}_2^{[2]} \end{bmatrix} \tag{A-1.9}$$

Where:

$\underline{y}_I^{[I]} = [a_x \ a_y \ a_z]^T$: is a 3X1 body acceleration vector.

$$[D_0] = \begin{bmatrix} 1 & 0 & 0 & 0 & d_z & -d_y \\ 0 & 1 & 0 & -d_z & 0 & d_x \\ 0 & 0 & 1 & d_y & -d_x & 0 \end{bmatrix} : \text{is a 3x6 accelerometer 'offset' matrix.}$$

$$[D_I] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & d_y & d_z & -d_x & 0 & -d_x \\ 0 & 1 & 0 & 0 & -1 & 0 & -d_y & d_x & 0 & 0 & d_z & -d_y \\ -1 & 0 & 1 & 0 & 0 & 0 & -d_z & 0 & d_x & -d_z & d_y & 0 \end{bmatrix} : \text{is a 3x12 accelerometer}$$

'offset' matrix.

Substituting from equation (A-1.5) for the $\frac{d}{dt}[\underline{\cdot}]$ term in the above equation gives us:

$$\begin{aligned}
\underline{y}_I^{[I]} &= [D_0 F_0 + D_I] \begin{bmatrix} \underline{x}_I^{[2]} \\ - \\ \underline{x}_2^{[2]} \end{bmatrix} + [D_0 G_0] \begin{bmatrix} \underline{u}_I^{[I]} \\ - \\ \underline{u}_2^{[I]} \end{bmatrix} + [D_0] \begin{bmatrix} \underline{g} \\ - \\ \underline{0} \end{bmatrix} \\
&\rightarrow \\
\underline{y}_I^{[I]} &= [D_0 F_0 + D_I] \underline{x}_3^{[2]} + [D_0 G_0] \underline{u}_3^{[I]} + [D_0] \underline{g}^{[I]}
\end{aligned} \tag{A-1.10}$$

Now, the expression for the body rates is:

$$\underline{y}_2^{[I]} = \underline{x}_2^{[I]} = [p \ q \ r]^T : \text{is a 3X1 body rate vector.}$$

Combining this with equation (A-1.10) gives us:

$$\begin{aligned} \begin{bmatrix} \underline{y}_1^{[I]} \\ \underline{y}_2^{[I]} \end{bmatrix} &= \begin{bmatrix} [0_{3 \times 6}] \\ [0_{3 \times 3} \quad / \quad I_{3 \times 3}] \end{bmatrix} \underline{x}_3^{[I]} + \begin{bmatrix} [D_0 F_0 + D_I] \\ [0_{3 \times 12}] \end{bmatrix} \underline{x}_3^{[2]} + \begin{bmatrix} [D_0 G_0] \\ [0_{3 \times 6}] \end{bmatrix} \underline{u}_3^{[I]} + \begin{bmatrix} [D_0] \\ [0_{3 \times 6}] \end{bmatrix} \underline{g}^{[I]} \\ \rightarrow \\ \underline{y}_3^{[I]} &= [J_0] \underline{x}_3^{[I]} + [K_0] \underline{x}_3^{[2]} + [L_0] \underline{u}_3^{[I]} + [M_0] \underline{g}^{[I]} \end{aligned} \quad (\text{A-1.11})$$

Where:

$$\underline{y}_3^{[I]} = \begin{bmatrix} \underline{y}_1^{[I]} \\ \underline{y}_2^{[I]} \end{bmatrix} : \text{is a } 6 \times 1 \text{ output vector.}$$

$$[J_0] = \begin{bmatrix} [0_{3 \times 6}] \\ [0_{3 \times 3} \quad / \quad I_{3 \times 3}] \end{bmatrix} : \text{is a } 6 \times 6 \text{ linear-state output coefficient matrix.}$$

$$[K_0] = \begin{bmatrix} [D_0 F_0 + D_I]_{3 \times 12} \\ [0_{3 \times 12}] \end{bmatrix} : \text{is a } 6 \times 12 \text{ quadratic-state output coefficient matrix.}$$

$$[L_0] = \begin{bmatrix} [D_0 G_0]_{3 \times 6} \\ [0_{3 \times 6}] \end{bmatrix} : \text{is a } 6 \times 6 \text{ coefficient matrix.}$$

$$[M_0] = \begin{bmatrix} [D_0]_{3 \times 6} \\ [0_{3 \times 6}] \end{bmatrix} : \text{is a } 6 \times 6 \text{ output coefficient matrix.}$$

Note: equation (A-1.11) represents the actual accelerations and body rates outputs; these have to be measured using a body fixed IMU (accelerometers and gyros).

A.1.2 Accelerometer and Gyro Dynamics

Let us assume a second order dynamics for the accelerometers and gyros respectively; thus:

$$\frac{a_{\alpha o}}{a_{\alpha}} = \frac{\omega_{\alpha}^2}{(s^2 + 2\zeta_{\alpha}\omega_{\alpha}s + \omega_{\alpha}^2)}; \quad \alpha = x, y, z \quad (\text{A-1.12})$$

$$\frac{\mu_o}{\mu} = \left(\frac{\omega_\mu^2}{s^2 + 2\zeta_\mu \omega_\mu s + \omega_\mu^2} \right); \quad \mu = p, q, r \quad (\text{A-1.13})$$

(ω, ζ) : denote the sensor natural frequency and the damping factor respectively. Subscript 'o' denote output values, and subscripts (x, y, z) and (p, q, r) denote accelerations and body rates measured by accelerometer and gyro orthogonal triads respectively.

A.1.3 Accelerometer Dynamics

In state space form equations for the accelerometer may be written as:

$$\frac{d}{dt} \begin{bmatrix} \dot{a}_{xo} \\ a_{xo} \\ \dot{a}_{yo} \\ a_{yo} \\ \dot{a}_{zo} \\ a_{zo} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_x^2 & -2\zeta_x \omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_y^2 & -2\zeta_y \omega_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & -2\zeta_z \omega_z \end{bmatrix} \begin{bmatrix} \dot{a}_{xo} \\ a_{xo} \\ \dot{a}_{yo} \\ a_{yo} \\ \dot{a}_{zo} \\ a_{zo} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_z^2 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad (\text{A-1.14})$$

Where: $\dot{a}_{\alpha o} = \frac{d}{dt} a_{\alpha o}; \quad \alpha = x, y, z$

This equation is of the form:

$$\frac{d}{dt} \underline{x}_4^{[I]} = [W_0] \underline{x}_4^{[I]} + [W_I] \underline{y}_I^{[I]} \quad (\text{A-1.15})$$

Where:

$$[W_0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_x^2 & -2\zeta_x \omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_y^2 & -2\zeta_y \omega_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & -2\zeta_z \omega_z \end{bmatrix} : \text{ is a } 6 \times 6 \text{ coefficient matrix}$$

containing accelerometer parameters.

$$[W_I] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_z^2 \end{bmatrix} : \text{is a } 6 \times 3 \text{ coefficient matrix containing accelerometer parameters.}$$

$$\underline{x}_4^{[I]} = \begin{bmatrix} a_{xo} & \dot{a}_{xo} & a_{yo} & \dot{a}_{yo} & a_{zo} & \dot{a}_{zo} \end{bmatrix}^T : \text{is a } 6 \times 1 \text{ accelerometer state vector.}$$

Substituting for $\underline{y}_1^{[I]}$ from equation (A-1.10) into equation (A-1.15), we get:

$$\frac{d}{dt} \underline{x}_4^{[I]} = [W_0] \underline{x}_4^{[I]} + [W_I D_0 F_0 + W_I D_I] \underline{x}_3^{[2]} + [W_I D_0 G_0] \underline{u}_3^{[I]} + [W_I D_0] \underline{g}^{[I]} \quad (\text{A-1.16})$$

→

$$\frac{d}{dt} \underline{x}_4^{[I]} = [W_0] \underline{x}_4^{[I]} + [W_2] \underline{x}_3^{[2]} + [W_3] \underline{u}_3^{[I]} + [W_4] \underline{g}^{[I]} \quad (\text{A-1.17})$$

Where:

$$[W_2] = [W_I D_0 F_0 + W_I D_I] : \text{is a } 6 \times 12 \text{ coefficient matrix.}$$

$$[W_3] = [W_I D_0 G_0] : \text{is a } 6 \times 6 \text{ coefficient matrix.}$$

$$[W_4] = [W_I D_0] : \text{is a } 6 \times 6 \text{ coefficient matrix.}$$

The measurement model is given by:

$$\underline{z}_4^{[I]} = [J_I] \underline{x}_4^{[I]} + \underline{v}_{a_b}^{[I]} + \underline{v}_{a_d}^{[I]} + \underline{v}_{a_s}^{[I]} + \underline{v}_{a_n}^{[I]} \quad (\text{A-1.18})$$

Where:

$$\underline{z}_4^{[I]} = \begin{bmatrix} a_{x_m} & a_{y_m} & a_{z_m} \end{bmatrix}^T : \text{is a } 3 \times 1 \text{ accelerometer measurement vector.}$$

$$\underline{v}_{a_b}^{[I]} = \begin{bmatrix} v_{x_b} & v_{y_b} & v_{z_b} \end{bmatrix}^T : \text{is a } 3 \times 1 \text{ accelerometer bias error vector.}$$

$$\underline{v}_{a_d}^{[I]} = \begin{bmatrix} v_{x_d} & v_{y_d} & v_{z_d} \end{bmatrix}^T : \text{is a } 3 \times 1 \text{ accelerometer drift error vector.}$$

$$\underline{v}_{a_s}^{[I]} = \begin{bmatrix} v_{x_s} & v_{y_s} & v_{z_s} \end{bmatrix}^T : \text{is a } 3 \times 1 \text{ accelerometer scale factor error vector.}$$

$\underline{v}_a^{[I]} = \begin{bmatrix} v_{x_n} & v_{y_n} & v_{z_n} \end{bmatrix}^T$: is a 3x1 accelerometer noise error vector.

$[J_I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$: is a 3x6 matrix.

A.1.4 Gyro Dynamics

Similarly, state space form equations for the gyros may be written as:

$$\frac{d}{dt} \begin{bmatrix} p_o \\ \dot{p}_o \\ q_o \\ \dot{q}_o \\ r_o \\ \dot{r}_o \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_p^2 & -2\zeta_p\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_q^2 & -2\zeta_q\omega_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_r^2 & -2\zeta_r\omega_r \end{bmatrix} \begin{bmatrix} p_o \\ \dot{p}_o \\ q_o \\ \dot{q}_o \\ r_o \\ \dot{r}_o \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_q^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (\text{A-1.19})$$

Where:

$$\dot{\mu}_o = \frac{d}{dt} \mu_o; \quad \mu = p, q, r$$

Equation (A-1.19) is of the form:

$$\frac{d}{dt} \underline{x}_5^{[I]} = [W_5] \underline{x}_5^{[I]} + [W_6] \underline{x}_2^{[I]} \quad (\text{A-1.20})$$

→

$$\frac{d}{dt} \underline{x}_5^{[I]} = [W_5] \underline{x}_5^{[I]} + [0_{6 \times 3} \quad W_6] \underline{x}_3^{[I]}$$

→

$$\frac{d}{dt} \underline{x}_5^{[I]} = [W_5] \underline{x}_5^{[I]} + [W_7] \underline{x}_3^{[I]} \quad (\text{A-1.21})$$

Where:

$\underline{x}_5^{[I]} = \begin{bmatrix} p_o & \dot{p}_o & q_o & \dot{q}_o & r_o & \dot{r}_o \end{bmatrix}^T$: is a 6x1 gyro state vector.

$$\underline{x}_2^{[I]} = \underline{y}_2^{[I]} = \underline{\omega} = \begin{bmatrix} p & q & r \end{bmatrix}^T$$

$$[W_5] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_p^2 & -2\zeta_p\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_q^2 & -2\zeta_q\omega_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_r^2 & -2\zeta_r\omega_r \end{bmatrix} : \text{ is a } 6 \times 6 \text{ coefficient matrix}$$

containing gyro parameters.

$$[W_6] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_q^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 \end{bmatrix} : \text{ is a } 6 \times 3 \text{ coefficient matrix containing gyro parameters.}$$

$$[W_7] = [0_{6 \times 3} \quad / \quad W_6] : \text{ is a } 6 \times 6 \text{ coefficient matrix.}$$

The measurement model is given by:

$$\underline{z}_5^{[I]} = [J_2] \underline{x}_5^{[I]} + \underline{v}_{g_b}^{[I]} + \underline{v}_{g_d}^{[I]} + \underline{v}_{g_s}^{[I]} + \underline{v}_{g_n}^{[I]} \quad (\text{A-1.22})$$

Where:

$$\underline{z}_4^{[I]} = \begin{bmatrix} p_m & q_m & r_m \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ gyro measurement vector.}$$

$$\underline{v}_{g_b}^{[I]} = \begin{bmatrix} v_{p_b} & v_{q_b} & v_{r_b} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ gyro bias error vector.}$$

$$\underline{v}_{g_d}^{[I]} = \begin{bmatrix} v_{p_d} & v_{q_d} & v_{r_d} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ gyro drift error vector.}$$

$$\underline{v}_{g_s}^{[I]} = \begin{bmatrix} v_{p_s} & v_{q_s} & v_{r_s} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ gyro scale factor error vector.}$$

$$\underline{v}_{g_n}^{[I]} = \begin{bmatrix} v_{p_n} & v_{q_n} & v_{r_n} \end{bmatrix}^T : \text{ is a } 3 \times 1 \text{ gyro noise error vector.}$$

$$[J_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} : \text{ is a } 3 \times 6 \text{ matrix.}$$

A.1.5 Actuation Servo Model

Let us assume a second order dynamics for the actuators, that is:

$$\frac{\alpha_o}{\alpha_i} = \left(\frac{\omega_\alpha^2}{s^2 + 2\zeta_\alpha \omega_\alpha s + \omega_\alpha^2} \right); \quad \alpha = \xi, \eta, \zeta \quad (\text{A-1.23})$$

$(\omega_\alpha, \zeta_\alpha)$: denote the sensor natural frequency and the damping factor respectively; subscript 'i' denotes the servo (control) input value (servo demand), and subscripts (ξ, η, ζ) denote roll, pitch, and yaw outputs from the servo.

In state space form equations for the actuation system model may be written as:

$$\frac{d}{dt} \begin{bmatrix} \xi_o \\ \dot{\xi}_o \\ \eta_o \\ \dot{\eta}_o \\ \zeta_o \\ \dot{\zeta}_o \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_\xi^2 & -2\zeta_\xi \omega_\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_\eta^2 & -2\zeta_\eta \omega_\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_\zeta^2 & -2\zeta_\zeta \omega_\zeta \end{bmatrix} \begin{bmatrix} \xi_o \\ \dot{\xi}_o \\ \eta_o \\ \dot{\eta}_o \\ \zeta_o \\ \dot{\zeta}_o \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega_\xi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_\eta^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_\zeta^2 \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{bmatrix} \quad (\text{A-1.24})$$

Where:

$$\dot{\alpha} = \frac{d}{dt} \alpha; \quad \alpha = \xi, \eta, \zeta$$

This equation is of the form:

$$\frac{d}{dt} \underline{x}_6^{[I]} = [\underline{V}_0] \underline{x}_6^{[I]} + [\underline{V}_I] \underline{u}_4^{[I]} \quad (\text{A-1.25})$$

Where:

$$\underline{x}_6^{[I]} = \begin{bmatrix} \xi_o & \dot{\xi}_o & \eta_o & \dot{\eta}_o & \zeta_o & \dot{\zeta}_o \end{bmatrix}^T : \text{ is a 6x1 state vector.}$$

$$\underline{u}_4^{[I]} = \underline{\alpha}_i = \begin{bmatrix} \xi_i & \eta_i & \zeta_i \end{bmatrix}^T : \text{ is a 3x1 control (servo actuator) input vector.}$$

$$[V_0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_\xi^2 & -2\zeta_\xi\omega_\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_\eta^2 & -2\zeta_\eta\omega_\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_\zeta^2 & -2\zeta_\zeta\omega_\zeta \end{bmatrix} : \text{is a 6x6 servo actuator coefficient matrix.}$$

$$[V_I] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_\xi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_\eta^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_\zeta^2 \end{bmatrix} : \text{is a 6x3 servo input coefficient matrix.}$$

If the actuator system noise is included in the model then the actual output from the actuator servo may be written as:

$$\frac{d}{dt} \underline{x}_6^{[I]} = [V_0] \underline{x}_6^{[I]} + [V_I] \underline{u}_4^{[I]} + \underline{v}_{s_n}^{[I]} \quad (\text{A-1.26})$$

We may also write for the actuator output:

$$\underline{y}_6^{[I]} = [\xi_o \quad \eta_o \quad \zeta_o]^T = [V_2] \underline{x}_6^{[I]} \quad (\text{A-1.27})$$

Where:

$$\underline{v}_{s_n}^{[I]} = [v_{\xi_n} \quad v_{\dot{\xi}_n} \quad v_{\eta_n} \quad v_{\dot{\eta}_n} \quad v_{\zeta_n} \quad v_{\dot{\zeta}_n}]^T : \text{is a 6x1 actuator servo noise error vector.}$$

$$[V_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} : \text{is a 3x6 matrix.}$$

Note that:

$$\begin{aligned} \underline{u}_3^{[I]} &= [\tilde{X}(\cdot) \quad \tilde{Y}(\cdot) \quad \tilde{Z}(\cdot) \quad L(\cdot) \quad M(\cdot) \quad N(\cdot)]^T \\ &= \underline{f}(u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi_o, \eta_o, \zeta_o) = \underline{f}(\underline{x}_3^{[I]}, \dot{\underline{x}}_3^{[I]}, [V_2] \underline{x}_6^{[I]}) \end{aligned} \quad (\text{A-1.28})$$

A.1.6 Non-linear Airframe, IMU and Actuator Dynamic

We can now combine equations (A-1.5), (A-1.17), (A-1.21) and (A-1.26) to give us an overall airframe, IMU and actuator dynamic model; this equation is of the Form:

$$\frac{d}{dt} \underline{x}_7^{[I]} = [F_1] \underline{x}_7^{[I]} + [F_2] \underline{x}_3^{[2]} + [G_1] \underline{u}_3^{[I]}(..) + [G_2] \underline{u}_4^{[I]} + [H_1] \underline{g}^{[I]} + [H_2] \underline{v}_{s_n}^{[I]} \quad (A-1.29)$$

Where:

$$\underline{x}_7^{[I]} = \left[\underline{x}_3^{[I]T} \quad / \quad \underline{x}_4^{[I]T} \quad / \quad \underline{x}_5^{[I]T} \quad / \quad \underline{x}_6^{[I]T} \right]^T : \text{is a } 24 \times 1 \text{ state vector.}$$

$$[F_1] = \begin{bmatrix} [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ --- & / & --- & / & --- & / & --- \\ [0_{6 \times 6}] & / & [W_0] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ --- & / & --- & / & --- & / & --- \\ [W_7] & / & [0_{6 \times 6}] & / & [W_5] & / & [0_{6 \times 6}] \\ --- & / & --- & / & --- & / & --- \\ [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [V_0] \end{bmatrix} : \text{is a } 24 \times 24 \text{ coefficient matrix.}$$

$$[F_2] = \left[[F_0]^T \quad / \quad [W_2]^T \quad / \quad [0_{12 \times 6}] \quad / \quad [0_{12 \times 6}] \right]^T : \text{is a } 24 \times 12 \text{ coefficient matrix.}$$

$$[G_1] = \left[[G_0]^T \quad / \quad [W_3]^T \quad / \quad [0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \right]^T : \text{is a } 24 \times 6 \text{ coefficient matrix.}$$

$$[G_2] = \left[[0_{3 \times 6}] \quad / \quad [0_{3 \times 6}] \quad / \quad [0_{3 \times 6}] \quad / \quad [V_1]^T \right]^T : \text{is a } 24 \times 3 \text{ coefficient matrix.}$$

$$[H_1] = \left[[I_{6 \times 6}] \quad / \quad [W_4]^T \quad / \quad [0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \right]^T : \text{is a } 24 \times 6 \text{ coefficient matrix.}$$

$$[H_2] = \left[[0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \quad / \quad [0_{6 \times 6}] \quad / \quad [I_{6 \times 6}] \right]^T : \text{is a } 24 \times 6 \text{ coefficient matrix.}$$

A block diagram of the decomposed version (derived by considering the sub-matrices) of the overall model is given in Figure A-1.1. See appendix section 3.

A.1.7 Measurement Model

Equations (A-1.18) and (A.1.22) may be combined to give the overall measurement model:

$$\underline{z}_7^{[I]} = [J_6] \underline{x}_7^{[I]} + \underline{v}_b^{[I]} + \underline{v}_d^{[I]} + \underline{v}_s^{[I]} + \underline{v}_n^{[I]} \quad (A-1.30)$$

Where:

$\underline{z}_7^T = \left[\underline{z}_4^T \quad / \quad \underline{z}_5^T \right]^T = \begin{bmatrix} a_{x_m} & a_{y_m} & a_{z_m} & p_m & q_m & r_m \end{bmatrix}^T$: is a 6x1 IMU (gyro + accelerometer) measurement vector.

$\underline{v}_b^T = \left[\underline{v}_{a_b}^T \quad / \quad \underline{v}_{g_b}^T \right]^T = \begin{bmatrix} v_{x_b} & v_{y_b} & v_{z_b} & v_{p_b} & v_{q_b} & v_{r_b} \end{bmatrix}^T$: is a 6x1 IMU bias error vector.

$\underline{v}_d^T = \left[\underline{v}_{a_d}^T \quad / \quad \underline{v}_{g_d}^T \right]^T = \begin{bmatrix} v_{x_d} & v_{y_d} & v_{z_d} & v_{p_d} & v_{q_d} & v_{r_d} \end{bmatrix}^T$: is a 6x1 IMU drift error vector.

$\underline{v}_s^T = \left[\underline{v}_{a_s}^T \quad / \quad \underline{v}_{g_s}^T \right]^T = \begin{bmatrix} v_{x_s} & v_{y_s} & v_{z_s} & v_{p_s} & v_{q_s} & v_{r_s} \end{bmatrix}^T$: is a 6x1 IMU scale factor error vector.

$\underline{v}_n^T = \left[\underline{v}_{a_n}^T \quad / \quad \underline{v}_{g_n}^T \right]^T = \begin{bmatrix} v_{x_n} & v_{y_n} & v_{z_n} & v_{p_n} & v_{q_n} & v_{r_n} \end{bmatrix}^T$: is a 6x1 IMU noise error vector.

$[J_6] = \begin{bmatrix} 0_{3 \times 6} & / & J_1 & / & 0_{3 \times 6} & / & 0_{3 \times 6} \\ --- & / & --- & / & --- & / & --- \\ 0_{3 \times 6} & / & 0_{3 \times 6} & / & J_2 & / & 0_{3 \times 6} \end{bmatrix}$: is a 6x24 matrix.

A.2. Linearised Airframe, Actuation and IMU Dynamics

Equation (A1.30) defines the complete non-linear description of the full 6-DOF airframe model. These equations contain quadratic terms in states and will be classed as the quadratic dynamic model. This type of model is required when autopilot design is undertaken for a missile executing high g- or high angle of attack manoeuvres, and (u, v, w, p, q, r) are not small. A more detailed consideration of the algebraic structure of this type of dynamic systems is given in [4].

A.2.1 Linearised Aerodynamic Forces and Moments

Assuming that $\tilde{X}, \tilde{Y}, \tilde{Z}, L, M$ and N are functions of $u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \dot{\xi}, \dot{\eta}, \dot{\zeta}$ and using first order linearisation about the nominal values $u_0, v_0, w_0, p_0, q_0, r_0, \xi_0, \eta_0$ and ζ_0 , we get:

$$\begin{aligned}
\Delta \underline{u}_3^{[I]} = & \begin{bmatrix} \Delta \tilde{X} + \Delta \tilde{T} \\ \Delta \tilde{Y} \\ \Delta \tilde{Z} \\ \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} \tilde{X}_u & \tilde{X}_v & \tilde{X}_w & \tilde{X}_p & \tilde{X}_q & \tilde{X}_r \\ \tilde{Y}_u & \tilde{Y}_v & \tilde{Y}_w & \tilde{Y}_p & \tilde{Y}_q & \tilde{Y}_r \\ \tilde{Z}_u & \tilde{Z}_v & \tilde{Z}_w & \tilde{Z}_p & \tilde{Z}_q & \tilde{Z}_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} + \begin{bmatrix} \tilde{X}_\xi & \tilde{X}_\eta & \tilde{X}_\varsigma \\ \tilde{Y}_\xi & \tilde{Y}_\eta & \tilde{Y}_\varsigma \\ \tilde{Z}_\xi & \tilde{Z}_\eta & \tilde{Z}_\varsigma \\ L_\xi & L_\eta & L_\varsigma \\ M_\xi & M_\eta & M_\varsigma \\ N_\xi & N_\eta & N_\varsigma \end{bmatrix} \begin{bmatrix} \Delta \xi_o \\ \Delta \eta_o \\ \Delta \varsigma_o \end{bmatrix} \\
& + \begin{bmatrix} \tilde{X}_{\dot{u}} & \tilde{X}_{\dot{v}} & \tilde{X}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Y}_{\dot{u}} & \tilde{Y}_{\dot{v}} & \tilde{Y}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Z}_{\dot{u}} & \tilde{Z}_{\dot{v}} & \tilde{Z}_{\dot{w}} & 0 & 0 & 0 \\ L_{\dot{u}} & L_{\dot{v}} & L_{\dot{w}} & 0 & 0 & 0 \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & 0 & 0 & 0 \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{v} \\ \Delta \dot{w} \\ \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{bmatrix}
\end{aligned} \tag{A-2.1}$$

This equation may be written as:

$$\begin{aligned}
\Delta \underline{u}_3^{[I]} &= [E_0] \Delta \underline{x}_3^{[I]} + [E_1 V_2] \Delta \underline{x}_6^{[I]} + [E_3] \frac{d}{dt} \Delta \underline{x}_3^{[I]} \\
\rightarrow \\
\Delta \underline{u}_3^{[I]} &= [E_0] \Delta \underline{x}_3^{[I]} + [E_2] \Delta \underline{x}_6^{[I]} + [E_3] \frac{d}{dt} \Delta \underline{x}_3^{[I]}
\end{aligned} \tag{A-2.2}$$

Where:

$\Delta(\cdot)$: denotes 'small' deviations from the normal steady-state condition (see Tables A-1.1 – A1-3):

$$\begin{aligned}
\tilde{X}_\alpha &= \frac{\partial \tilde{X}}{\partial \alpha}; \quad \alpha = u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi, \eta, \varsigma; \quad \tilde{Y}_\alpha = \frac{\partial \tilde{Y}}{\partial \alpha}; \quad \alpha = u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi, \eta, \varsigma \\
\tilde{Z}_\alpha &= \frac{\partial \tilde{Z}}{\partial \alpha}; \quad \alpha = u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi, \eta, \varsigma; \quad L_\alpha = \frac{\partial L}{\partial \alpha}; \quad \alpha = u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi, \eta, \varsigma \\
M_\alpha &= \frac{\partial M}{\partial \alpha}; \quad \alpha = u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi, \eta, \varsigma; \quad N_\alpha = \frac{\partial N}{\partial \alpha}; \quad \alpha = u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \xi, \eta, \varsigma
\end{aligned}$$

$[E_2] = [E_1 V_2]$: is a 6x6 matrix.

$$\Delta \tilde{T} = 0; \quad \Delta \underline{g}^{[I]} = \underline{0}$$

$$[E_0] = \begin{bmatrix} \tilde{X}_u & \tilde{X}_v & \tilde{X}_w & \tilde{X}_p & \tilde{X}_q & \tilde{X}_r \\ \tilde{Y}_u & \tilde{Y}_v & \tilde{Y}_w & \tilde{Y}_p & \tilde{Y}_q & \tilde{Y}_r \\ \tilde{Z}_u & \tilde{Z}_v & \tilde{Z}_w & \tilde{Z}_p & \tilde{Z}_q & \tilde{Z}_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} : \text{is a 6x6 aero-derivative matrix.}$$

$$[E_I] = \begin{bmatrix} \tilde{X}_\xi & \tilde{X}_\eta & \tilde{X}_\varsigma \\ \tilde{Y}_\xi & \tilde{Y}_\eta & \tilde{Y}_\varsigma \\ \tilde{Z}_\xi & \tilde{Z}_\eta & \tilde{Z}_\varsigma \\ L_\xi & L_\eta & L_\varsigma \\ M_\xi & M_\eta & M_\varsigma \\ N_\xi & N_\eta & N_\varsigma \end{bmatrix} : \text{is a 6x3 control-derivative matrix.}$$

$$[E_3] = \begin{bmatrix} \tilde{X}_{\dot{u}} & \tilde{X}_{\dot{v}} & \tilde{X}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Y}_{\dot{u}} & \tilde{Y}_{\dot{v}} & \tilde{Y}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Z}_{\dot{u}} & \tilde{Z}_{\dot{v}} & \tilde{Z}_{\dot{w}} & 0 & 0 & 0 \\ L_{\dot{u}} & L_{\dot{v}} & L_{\dot{w}} & 0 & 0 & 0 \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & 0 & 0 & 0 \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & 0 & 0 & 0 \end{bmatrix} : \text{is a 6x6 aero-derivative matrix.}$$

A.2.2 Linearisation of the Quadratic State Vector

It is easily verified the first order linearization of the quadratic state vector defined in section 2.1. :

$$\underline{x}_3^{[2]} = \left[\underline{x}_I^{[2]T} \quad / \quad \underline{x}_2^{[2]T} \right]^T = \left[uq \quad ur \quad vp \quad vr \quad wp \quad wq \quad / \quad p^2 \quad pq \quad pr \quad q^2 \quad qr \quad r^2 \right]^T$$

may be written as:

$$\Delta \underline{x}_3^{[2]} = [E_4] \Delta \underline{x}_3^{[I]} \quad (\text{A-2.3})$$

Where:

$$[E_4] = \begin{bmatrix} q_0 & 0 & 0 & 0 & u_0 & 0 \\ r_0 & 0 & 0 & 0 & 0 & u_0 \\ 0 & p_0 & 0 & v_0 & 0 & 0 \\ 0 & r_0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & p_0 & w_0 & 0 & 0 \\ 0 & 0 & q_0 & 0 & w_0 & 0 \\ 0 & 0 & 0 & 2p_0 & 0 & 0 \\ 0 & 0 & 0 & q_0 & p_0 & 0 \\ 0 & 0 & 0 & r_0 & 0 & p_0 \\ 0 & 0 & 0 & 0 & 2q_0 & 0 \\ 0 & 0 & 0 & 0 & r_0 & q_0 \\ 0 & 0 & 0 & 0 & 0 & 2r_0 \end{bmatrix} : \text{is a } 12 \times 6 \text{ matrix of steady state values.}$$

$$\Delta \underline{x}_3^{[I]} = \left[\Delta \underline{x}_1^{[I]T} \quad / \quad \Delta \underline{x}_2^{[I]T} \right]^T = [\Delta u \quad \Delta v \quad \Delta w \quad / \quad \Delta p \quad \Delta q \quad \Delta r]^T : \text{is a } 6 \times 1 \text{ linear } \Delta \text{-state vector.}$$

A.2.3 Linearised Airframe, IMU, Actuator Model

The ‘small’ perturbation model for the quadratic dynamic model of equation (A-1.29) may be written as:

$$\frac{d}{dt} \Delta \underline{x}_7^{[I]} = [F_1] \Delta \underline{x}_7^{[I]} + [F_2] \Delta \underline{x}_3^{[2]} + [G_1] \Delta \underline{u}_3^{[I]} + [G_2] \Delta \underline{u}_4^{[I]} + [H_1] \Delta \underline{g}^{[I]} + [H_2] \Delta \underline{v}_{s_n}^{[I]} \quad (\text{A-2.4})$$

Substituting for $\Delta \underline{x}_3^{[2]}$ and $\Delta \underline{u}_3^{[I]}$ from equations (A-2.2) and (A-2.3), gives us:

$$\begin{aligned} \frac{d}{dt} \Delta \underline{x}_7^{[I]} = & [F_1] \Delta \underline{x}_7^{[I]} + [F_2 E_4 + G_1 E_0] \Delta \underline{x}_3^{[I]} + [G_1 E_2] \Delta \underline{x}_6^{[I]} + [G_1 E_3] \frac{d}{dt} \Delta \underline{x}_3^{[I]} \\ & + [G_2] \Delta \underline{u}_4^{[I]} + [H_2] \Delta \underline{v}_{s_n}^{[I]} \end{aligned} \quad (\text{A-2.5})$$

Where:

$$\Delta \underline{g}^{[I]} = \underline{0}; [F_2 E_4 + G_1 E_0] = \begin{bmatrix} [F_0 E_4 + G_0 E_0] \\ \text{-----} \\ [W_2 E_4 + W_3 E_0] \\ \text{-----} \\ [0_{6 \times 6}] \\ \text{-----} \\ [0_{6 \times 6}] \end{bmatrix}; [G_1 E_2] = \begin{bmatrix} [G_0 E_2] \\ \text{-----} \\ [W_3 E_2] \\ \text{-----} \\ [0_{6 \times 6}] \\ \text{-----} \\ [0_{6 \times 6}] \end{bmatrix}; [G_1 E_3] = \begin{bmatrix} [G_0 E_3] \\ \text{-----} \\ [W_3 E_3] \\ \text{-----} \\ [0_{6 \times 6}] \\ \text{-----} \\ [0_{6 \times 6}] \end{bmatrix}$$

These above matrices are all of dimension 24X6. See also the block diagram Figure A-1.2.
Now, Equation (A-2.5) →

$$\begin{aligned} \frac{d}{dt} \Delta \underline{x}_7^{[I]} &= [F_1] \Delta \underline{x}_7^{[I]} + [[F_2 E_4 + G_1 E_0] \ / \ [0_{24 \times 6}] \ / \ [0_{24 \times 6}] \ / \ [G_1 E_2]] \Delta \underline{x}_7^{[I]} \\ &\quad + [G_2] \Delta \underline{u}_4^{[I]} + [[G_1 E_3] \ / \ [0_{24 \times 6}] \ / \ [0_{24 \times 6}] \ / \ [0_{24 \times 6}]] \frac{d}{dt} \Delta \underline{x}_7^{[I]} \\ &\quad + [H_2] \Delta \underline{v}_{s_n}^{[I]} \end{aligned} \quad (A-2.6)$$

→

$$\frac{d}{dt} [F_3] \Delta \underline{x}_7^{[I]} = [F_4] \Delta \underline{x}_7^{[I]} + [G_2] \Delta \underline{u}_4^{[I]} + [H_2] \Delta \underline{v}_{s_n}^{[I]} \quad (A-2.7)$$

→

$$\frac{d}{dt} \Delta \underline{x}_6^{[I]} = [F_3]^{-I} [F_4] \Delta \underline{x}_7^{[I]} + [F_3]^{-I} [G_2] \Delta \underline{u}_4^{[I]} + [F_3]^{-I} [H_2] \Delta \underline{v}_{s_n}^{[I]} \quad (A-2.8)$$

→

$$\frac{d}{dt} \Delta \underline{x}_7^{[I]} = [F_5] \Delta \underline{x}_7^{[I]} + [G_3] \Delta \underline{u}_4^{[I]} + [H_3] \Delta \underline{v}_{s_n}^{[I]} \quad (A-2.9)$$

Where:

$[F_3] = [I_{24 \times 24}] - [G_1 E_3 \ / \ 0_{24 \times 6} \ / \ 0_{24 \times 6} \ / \ 0_{24 \times 6}]$: is a 24x24 coefficient matrix.

$[F_4] = [F_1] + [F_2 E_4 + G_1 E_0 \ / \ 0_{24 \times 6} \ / \ 0_{24 \times 6} \ / \ G_1 E_2]$: is a 24x24 coefficient matrix.

$[F_5] = [F_3]^{-I} [F_4]$: is a 24x24 coefficient matrix.

$[G_3] = [F_3]^{-I} [G_2]$: is a 24x3 coefficient matrix.

$[H_3] = [F_3]^{-I} [H_2]$: is a 24x3 coefficient matrix.

Small perturbation model of the measurement model (see equation (A-1.30) may be written as:

$$\Delta \underline{z}_7^{[I]} = [J_6] \Delta \underline{x}_7^{[I]} + \Delta \underline{v}_b^{[I]} + \Delta \underline{v}_b^{[I]} + \Delta \underline{v}_s^{[I]} + \Delta \underline{v}_n^{[I]} \quad (A-2.10)$$

Δ : denotes small perturbation about nominal values.

Finally, linearising the output equation (A-1.11) gives us:

$$\underline{\Delta y}_3^{[I]} = [J_0 + K_0 E_4] \underline{\Delta x}_3^{[I]} + [L_0] \underline{\Delta u}_3^{[I]} \quad (\text{A-2.11})$$

→

$$\underline{\Delta y}_3^{[I]} = [J_7] \underline{\Delta x}_3^{[I]} + [L_0] \underline{\Delta u}_3^{[I]} \quad (\text{A-2.12})$$

Where:

$[J_7] = [J_0 + K_0 E_4]$: is a 6x6 coefficient matrix.

A.3. Decomposed State Space Models

A.3.1 Nonlinear Model:

Using the relationship established in the appendix section 1.4, we may write:

$$\frac{d}{dt} \begin{bmatrix} \underline{x}_3^{[I]} \\ \underline{x}_4^{[I]} \\ \underline{x}_5^{[I]} \\ \underline{x}_6^{[I]} \end{bmatrix} = \begin{bmatrix} [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ [0_{6 \times 6}] & / & [W_0] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ [W_7] & / & [0_{6 \times 6}] & / & [W_5] & / & [0_{6 \times 6}] \\ [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [V_0] \end{bmatrix} \begin{bmatrix} \underline{x}_3^{[I]} \\ \underline{x}_4^{[I]} \\ \underline{x}_5^{[I]} \\ \underline{x}_6^{[I]} \end{bmatrix} + \begin{bmatrix} [F_0] \\ [W_2] \\ [0_{6 \times 12}] \\ [0_{6 \times 12}] \end{bmatrix} \underline{x}_3^{[2]} + \begin{bmatrix} [G_0] \\ [W_3] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \end{bmatrix} \underline{u}_3^{[I]} + \begin{bmatrix} [0_{6 \times 3}] \\ [0_{6 \times 3}] \\ [0_{6 \times 3}] \\ [V_1] \end{bmatrix} \underline{u}_4^{[I]} + \begin{bmatrix} [I_{6 \times 6}] \\ [W_4] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \end{bmatrix} \underline{g}^{[I]} + \begin{bmatrix} [0_{6 \times 6}] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \\ [I_{6 \times 6}] \end{bmatrix} \underline{v}_{s_n}^{[I]} \quad (A3-1)$$

A.3.2 Linearised Model:

Using the relationship established in the appendix section 1.4 and equations (A-2.2), (A-2.3) and (A-2.5), we may write:

$$\frac{d}{dt} \begin{bmatrix} \underline{\Delta x}_3^{[I]} \\ \underline{\Delta x}_4^{[I]} \\ \underline{\Delta x}_5^{[I]} \\ \underline{\Delta x}_6^{[I]} \end{bmatrix} = \begin{bmatrix} [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ [0_{6 \times 6}] & / & [W_0] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] \\ [W_7] & / & [0_{6 \times 6}] & / & [W_5] & / & [0_{6 \times 6}] \\ [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [0_{6 \times 6}] & / & [V_0] \end{bmatrix} \begin{bmatrix} \underline{\Delta x}_3^{[I]} \\ \underline{\Delta x}_4^{[I]} \\ \underline{\Delta x}_5^{[I]} \\ \underline{\Delta x}_6^{[I]} \end{bmatrix} + \begin{bmatrix} [F_0 E_4 + G_0 E_0] \\ [W_2 E_4 + W_3 E_0] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \end{bmatrix} \underline{\Delta x}_3^{[I]} + \begin{bmatrix} [G_0 E_2] \\ [W_3 E_2] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \end{bmatrix} \underline{\Delta x}_6^{[I]} + \begin{bmatrix} [0_{6 \times 3}] \\ [0_{6 \times 3}] \\ [0_{6 \times 3}] \\ [V_1] \end{bmatrix} \underline{\Delta u}_4^{[I]} + \begin{bmatrix} [G_0 E_3] \\ [W_3 E_3] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \end{bmatrix} \frac{d}{dt} \underline{\Delta x}_3^{[I]} + \begin{bmatrix} [0_{6 \times 6}] \\ [0_{6 \times 6}] \\ [0_{6 \times 6}] \\ [I_{6 \times 6}] \end{bmatrix} \underline{\Delta v}_{s_n}^{[I]} \quad (A3-2)$$

Block diagrams for the decomposed versions for the nonlinear and the linearised models are given in Figures A-1.1 and A-1.2 respectively

Table A-1.1. Longitudinal (Roll) Aerodynamic Derivatives and Coefficients:

Derivatives		Derivatives		Normalised Coefficients	
Symbol	Units	Symbols	Units	Symbols	Units
X_u	$Nm^{-1} sec$	-	-	$C_{x_u} (= X_u / QS_x)$	$m^{-1} sec$
X_v	$Nm^{-1} sec$	$X_{\beta} (= UX_v)$	N	$C_{x_{\beta}} (= X_{\beta} / QS_x)$	-
X_w	$Nm^{-1} sec$	$X_{\alpha} (= UX_w)$	N	$C_{x_{\alpha}} (= X_{\alpha} / QS_x)$	-
X_p	$N sec$	-	-	$C_{x_p} (= X_p / QS_x)$	sec
X_q	$N sec$	-	-	$C_{x_q} (= X_q / QS_x)$	sec
X_r	$N sec$	-	-	$C_{x_r} (= X_r / QS_x)$	sec
X_{ξ}	N	-	-	$C_{x_{\xi}} (= X_{\xi} / QS_x)$	-
X_{η}	N	-	-	$C_{x_{\eta}} (= X_{\eta} / QS_x)$	-
X_{ζ}	N	-	-	$C_{x_{\zeta}} (= X_{\zeta} / QS_x)$	-
$X_{\ddot{u}}$	$Nm^{-1} sec^2$	-	-	$C_{x_{\ddot{u}}} (= X_{\ddot{u}} / QS_x)$	$m^{-1} sec^2$
$X_{\dot{v}}$	$Nm^{-1} sec^2$	$X_{\dot{\beta}} (= UX_{\dot{v}})$	$N sec$	$C_{x_{\dot{v}}} (= X_{\dot{v}} / QS_x)$	$m^{-1} sec^2$
$X_{\dot{w}}$	$Nm^{-1} sec^2$	$X_{\dot{\alpha}} (= UX_{\dot{w}})$	$N sec$	$C_{x_{\dot{w}}} (= X_{\dot{w}} / QS_x)$	$m^{-1} sec^2$
L_u	$N sec$	-	-	$C_{l_u} (= L_u / QS_x c_x)$	$m^{-1} sec$
L_v	$N sec$	-	-	$C_{l_v} (= L_v / QS_x c_x)$	$m^{-1} sec$
L_w	$N sec$	-	-	$C_{l_w} (= L_w / QS_x c_x)$	$m^{-1} sec$
L_p	$Nm sec$	-	-	$C_{l_p} (= L_p / QS_x c_x)$	sec
L_q	$Nm sec$	-	-	$C_{l_q} (= L_q / QS_x c_x)$	sec
L_r	$Nm sec$	-	-	$C_{l_r} (= L_r / QS_x c_x)$	sec
L_{ξ}	Nm	-	-	$C_{l_{\xi}} (= L_{\xi} / QS_x c_x)$	-
L_{η}	Nm	-	-	$C_{l_{\eta}} (= L_{\eta} / QS_x c_x)$	-
L_{ζ}	Nm	-	-	$C_{l_{\zeta}} (= L_{\zeta} / QS_x c_x)$	-
$L_{\ddot{u}}$	$N sec^2$	-	-	$C_{l_{\ddot{u}}} (= L_{\ddot{u}} / QS_x c_x)$	$m^{-1} sec^2$
$L_{\dot{v}}$	$N sec^2$	$L_{\dot{\beta}} (= UL_{\dot{v}})$	$Nm sec$	$C_{l_{\dot{v}}} (= L_{\dot{v}} / QS_x c_x)$	$m^{-1} sec^2$
$L_{\dot{w}}$	$N sec^2$	$L_{\dot{\alpha}} (= UL_{\dot{w}})$	$Nm sec$	$C_{l_{\dot{w}}} (= L_{\dot{w}} / QS_x c_x)$	$m^{-1} sec^2$

$$* Q = \frac{1}{2} \rho U^2 : (Nm^{-2}); S_x : (m^2); c_x : (m)$$

Table A-2.1. Lateral (Pitch) Aerodynamic Derivatives and Coefficients:

Derivatives		Derivatives		Normalised Coefficients	
Symbol	Units	Symbols	Units	Symbols	Units
Y_u	$Nm^{-1} sec$	-	-	$C_{y_u} (= Y_u / QS_y)$	$m^{-1} sec$
Y_v	$Nm^{-1} sec$	$Y_{\beta} (= UY_v)$	N	$C_{y_{\beta}} (= Y_{\beta} / QS_y)$	-
Y_w	$Nm^{-1} sec$	$Y_{\alpha} (= UY_w)$	N	$C_{Y_{\alpha}} (= Y_{\alpha} / QS_y)$	-
Y_p	$N sec$	-	-	$C_{y_p} (= Y_p / QS_y)$	sec
Y_q	$N sec$	-	-	$C_{y_q} (= Y_q / QS_y)$	sec
Y_r	$N sec$	-	-	$C_{y_r} (= Y_r / QS_y)$	sec
Y_{ξ}	N	-	-	$C_{y_{\xi}} (= Y_{\xi} / QS_y)$	-
Y_{η}	N	-	-	$C_{y_{\eta}} (= Y_{\eta} / QS_y)$	-
Y_{ζ}	N	-	-	$C_{y_{\zeta}} (= Y_{\zeta} / QS_y)$	-
$Y_{\dot{u}}$	$Nm^{-1} sec^2$	-	-	$C_{y_{\dot{u}}} (= Y_{\dot{u}} / QS_y)$	$m^{-1} sec^2$
$Y_{\dot{v}}$	$Nm^{-1} sec^2$	$Y_{\dot{\beta}} (= UY_{\dot{v}})$	$N sec$	$C_{y_{\dot{v}}} (= Y_{\dot{v}} / QS_y)$	$m^{-1} sec^2$
$Y_{\dot{w}}$	$Nm^{-1} sec^2$	$Y_{\dot{\alpha}} (= UY_{\dot{w}})$	$N sec$	$C_{y_{\dot{w}}} (= Y_{\dot{w}} / QS_y)$	$m^{-1} sec^2$
M_u	$N sec$	-	-	$C_{m_u} (= M_u / QS_y c_y)$	$m^{-1} sec$
M_v	$N sec$	-	-	$C_{m_v} (= M_v / QS_y c_y)$	$m^{-1} sec$
M_w	$N sec$	-	-	$C_{m_w} (= M_w / QS_y c_y)$	$m^{-1} sec$
M_p	$Nm sec$	-	-	$C_{m_p} (= M_p / QS_y c_y)$	sec
M_q	$Nm sec$	-	-	$C_{m_q} (= M_q / QS_y c_y)$	sec
M_r	$Nm sec$	-	-	$C_{m_r} (= M_r / QS_y c_y)$	sec
M_{ξ}	Nm	-	-	$C_{m_{\xi}} (= M_{\xi} / QS_y c_y)$	-
M_{η}	Nm	-	-	$C_{m_{\eta}} (= M_{\eta} / QS_y c_y)$	-
M_{ζ}	Nm	-	-	$C_{m_{\zeta}} (= M_{\zeta} / QS_y c_y)$	-
$M_{\dot{u}}$	$N sec^2$	-	-	$C_{m_{\dot{u}}} (= M_{\dot{u}} / QS_y c_y)$	$m^{-1} sec^2$
$M_{\dot{v}}$	$N sec^2$	$M_{\dot{\beta}} (= UM_{\dot{v}})$	$Nm sec$	$C_{m_{\dot{v}}} (= M_{\dot{v}} / QS_y c_y)$	$m^{-1} sec^2$
$M_{\dot{w}}$	$N sec^2$	$M_{\dot{\alpha}} (= UM_{\dot{w}})$	$Nm sec$	$C_{m_{\dot{w}}} (= M_{\dot{w}} / QS_y c_y)$	$m^{-1} sec^2$

Table A-3.1. Lateral (Yaw) Aerodynamic Derivatives and Coefficients:

Derivatives		Derivatives		Normalised Coefficients	
Symbol	Units	Symbols	Units	Symbols	Units
Z_u	$Nm^{-1} sec$	-	-	$C_{z_u} (= Z_u / QS_z)$	$m^{-1} sec$
Z_v	$Nm^{-1} sec$	$Z_{\beta} (= UZ_v)$	N	$C_{z_{\beta}} (= Z_{\beta} / QS_z)$	-
Z_w	$Nm^{-1} sec$	$Z_{\alpha} (= UZ_w)$	N	$C_{z_{\alpha}} (= Z_{\alpha} / QS_z)$	-
Z_p	$N sec$	-	-	$C_{z_p} (= Z_p / QS_z)$	sec
Z_q	$N sec$	-	-	$C_{z_q} (= Z_q / QS_z)$	sec
Z_r	$N sec$	-	-	$C_{z_r} (= Z_r / QS_z)$	sec
Z_{ξ}	N	-	-	$C_{z_{\xi}} (= Z_{\xi} / QS_z)$	-
Z_{η}	N	-	-	$C_{z_{\eta}} (= Z_{\eta} / QS_z)$	-
Z_{ζ}	N	-	-	$C_{z_{\zeta}} (= Z_{\zeta} / QS_z)$	-
$Z_{\dot{u}}$	$Nm^{-1} sec^2$	-	-	$C_{z_{\dot{u}}} (= Z_{\dot{u}} / QS_z)$	$m^{-1} sec^2$
$Z_{\dot{v}}$	$Nm^{-1} sec^2$	$Z_{\dot{\beta}} (= UZ_{\dot{v}})$	$N sec$	$C_{z_{\dot{v}}} (= Z_{\dot{v}} / QS_z)$	$m^{-1} sec^2$
$Z_{\dot{w}}$	$Nm^{-1} sec^2$	$Z_{\dot{\alpha}} (= UZ_{\dot{w}})$	$N sec$	$C_{z_{\dot{w}}} (= Z_{\dot{w}} / QS_z)$	$m^{-1} sec^2$
N_u	$N sec$	-	-	$C_{n_u} (= N_u / QS_z c_z)$	$m^{-1} sec$
N_v	$N sec$	-	-	$C_{n_v} (= N_v / QS_z c_z)$	$m^{-1} sec$
N_w	$N sec$	-	-	$C_{n_w} (= N_w / QS_z c_z)$	$m^{-1} sec$
N_p	$Nm sec$	-	-	$C_{n_p} (= N_p / QS_z c_z)$	sec
N_q	$Nm sec$	-	-	$C_{n_q} (= N_q / QS_z c_z)$	sec
N_r	$Nm sec$	-	-	$C_{n_r} (= N_r / QS_z c_z)$	sec
N_{ξ}	Nm	-	-	$C_{n_{\xi}} (= N_{\xi} / QS_z c_z)$	-
N_{η}	Nm	-	-	$C_{n_{\eta}} (= N_{\eta} / QS_z c_z)$	-
N_{ζ}	Nm	-	-	$C_{n_{\zeta}} (= N_{\zeta} / QS_z c_z)$	-
$N_{\dot{u}}$	$N sec^2$	-	-	$C_{n_{\dot{u}}} (= N_{\dot{u}} / QS_z c_z)$	$m^{-1} sec^2$
$N_{\dot{v}}$	$N sec^2$	$N_{\dot{\beta}} (= UN_{\dot{v}})$	$Nm sec$	$C_{n_{\dot{v}}} (= N_{\dot{v}} / QS_z c_z)$	$m^{-1} sec^2$
$N_{\dot{w}}$	$N sec^2$	$N_{\dot{\alpha}} (= UL_{\dot{w}})$	$Nm sec$	$C_{n_{\dot{w}}} (= N_{\dot{w}} / QS_z c_z)$	$m^{-1} sec^2$

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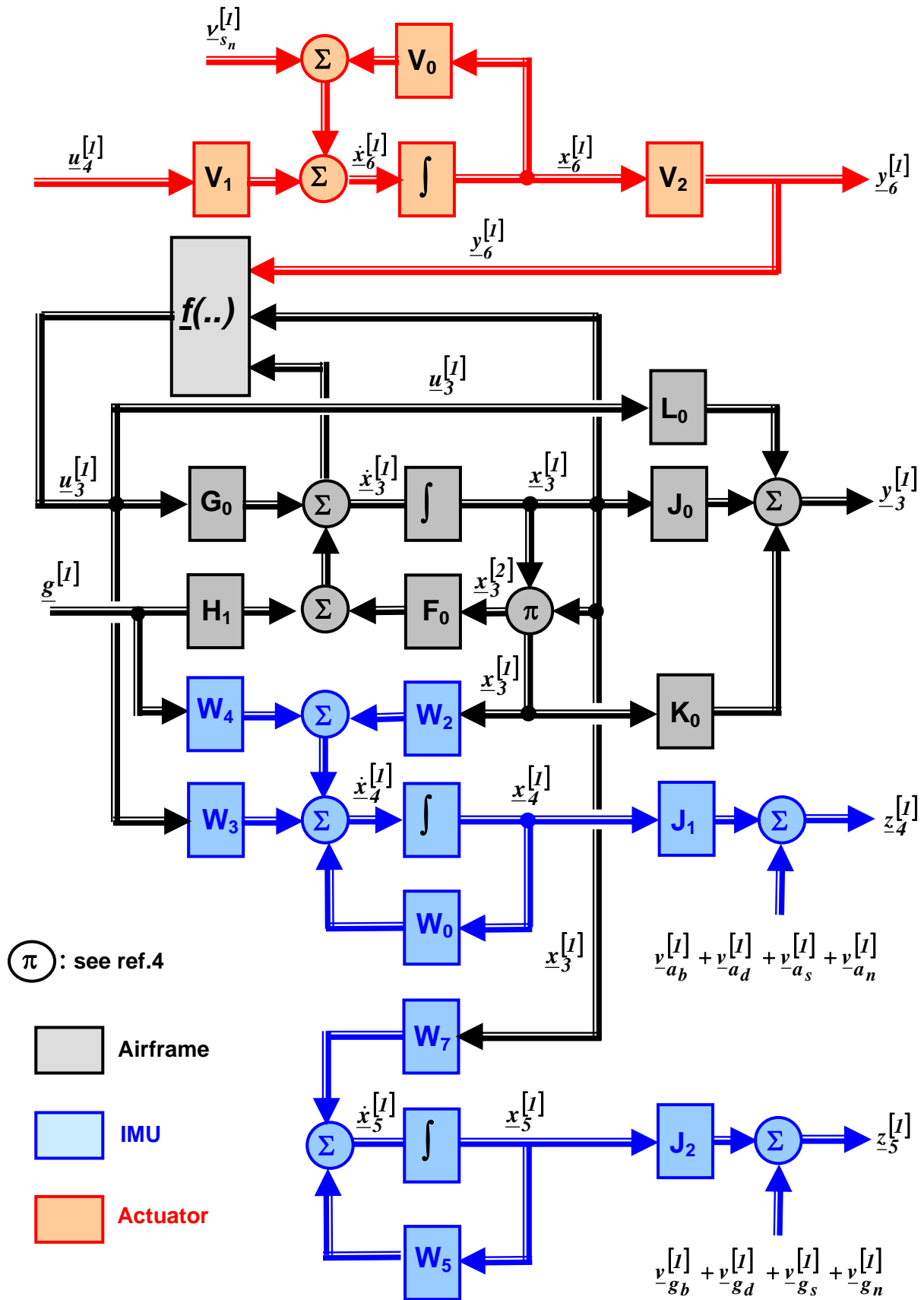


Figure A-1.1 Nonlinear (Quadratic) Airframe Model including Actuator and IMU

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19. ABSTRACT <p>This report is a follow on to the report given in DSTO-TN-0559 and considers the derivation of the mathematical model for a aerospace vehicles and missile autopilots in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the non-linearities are "structured" (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques, particularly for autopilot design of aerospace vehicles executing high g-maneuvres.</p> <p>This report also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. The models developed are useful for further research on precision optimum guidance and control. It is hoped that the model will provide more accurate presentations of missile autopilot dynamics and will be used for adaptive and integrated guidance & control of agile missiles.</p>					